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THE UNIVERSITY OF ALBERTA

CREATIVE PROBLEM SOLVING IN JUNIOR HIGH SCHOOL MATHEMATICS

by

© HALIA K. BOYCHUK

A THESIS

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FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Creative Problem Solving in Junior High School Mathematics submitted by Halia K. Boychuk in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

The purpose of the study was to describe creative thinking in mathematics as done by grade nine students. This purpose was carried out in three phases.

The first phase was to establish a model of creative problem solving in terms of four processes--two divergent (conjecturing, sensitivity) and two convergent (redefinition, verification). The model was defined by reference to literature, especially that of Guilford, Torrance, Polya, and Taylor-Pearce. Implicit in the definition of the model was the formulation of specific guidelines for problem-situations which reflected each process. Furthermore these guidelines were used to construct two specific problems for each process.

During the second phase of the study the eight problems were administered to forty-two grade nine students from four junior high schools in the Edmonton Public School System. The children were asked for written responses, oral elaborations on their responses, as well as comments on how they achieved their solutions. These oral responses were taped. Scores on standard measures of creativity (Torrance), achievement, and ability were also collected for each student. The written responses were scored, the scores were correlated and an image analysis procedure performed on the correlations. The correlational and the image analysis results were examined in an attempt to identify the constructs presented in the questions. This was done by discussing the extent to which the different scores on the same problem correlated, the scores on different problems for the same process correlated, the scores on the different processes correlated, the scores on the constructed problems correlated with the scores on the standard measures and the scores on the constructed problems clustered into groups identifiable by reference to the four processes presented in the hypothesized model.

The ~~third~~ phase centered about the students' oral elaborations on their solutions to the conjecturing problems. An attempt was made to classify responses according to how conjectures were made, and according to the levels of abstraction at which they were made. Specific responses for each classification are reported.

In conclusion, it was found that the image analysis on the correlations between scores on the problems separated into three factors which indicated the processes of conjecturing, redefinition, and verification. The two sensitivity problems loaded on two factors which separated from each other and from the other factors.

It was also found that the scores on the constructed tests did not correlate with the scores on the Torrance creativity tests. It seemed that these problems identified a facet of creativity not measured by the Torrance tests. The scores on the verification and redefinition problems correlated with the scores on achievement and ability. The problems were also found to intercorrelate in various ways. This seems to be due to the fact that each problem requires an interaction between several processes and is not a clear representation of only one process.

The analysis on the correlations and the examination of the oral responses did not discredit the model. It seems that the

responses of children to problems can be discussed in terms of conjecturing, sensitivity, redefinition and verification.

The data gathered in the study would seem to imply that further research into the types of thinking patterns used by students to solve problems can provide information useful to the classroom teacher in helping children learn mathematics.

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CHAPTER I

BACKGROUND AND SIGNIFICANCE OF THE STUDY

INTRODUCTION TO THE PROBLEM

Creativity has been hypothesized to exist in many forms, and to vary in requirement and form from one specific discipline to another. In mathematics curriculum, as in other curricula, new developments have emphasized the central role of the student. The interest in the discovery method in relation to mathematics teaching (Davis, Polya, Kemeny (1961), Hohn (1961), Hendrix (1961)) has resulted from the notion that students should be able to develop their own intuitive ideas of mathematical concepts and to explore relationships of their own formulation. Papert (1972) extends this notion even further. He states that the child is "constantly engaged in inventing theories about everything" (p. 3) and should be taught about thinking. He defends the right of the child to formulate theories and to test, verify or discard them, and maintains that children working on his computer programs have shown their capability to do this. Wheeler (1970) discusses the interest existing in Great Britain in establishing school mathematics as an inventive study, "To see mathematics as an invention rather than discovery and as applicable to virtually any raw material whatever can give us a different sense of its place in education." (p. 150) He defends the applicability of this view to general education maintaining that higher mental behaviors such as substitutions, reversals, and combinations have been observed in the spontaneous behaviour of pre-school children. "The illumination that these scientific observations can give is that the young child is already applying to data provided by his experience mental powers which are the basic constituents of the mathematicians' algebra. The revolution in teaching mathematics will come if and only if these powers are acknowledged as belonging to the child before anyone starts to teach him mathematics at all." (Wheeler, p. 150)

Banwell, Tahta, and Saunders (1972), who describe a number of mathematical starting points for projects which have been presented and successful with children, describe the importance of mathematics as problem-solving. Emphasized are the place of the refinement, the definition of a broad problem in mathematical problem-solving: "The importance of mathematics in education lies in process rather than product." (Banwell, p. 6)

These views maintain that school children are to some degree capable of mental processes of the higher forms. The International Study of Achievement in Mathematics (Husen, 1967, p. 81) when comparing mathematics curricula in twelve countries throughout the world, listed "Inventiveness: reasoning creativity in mathematics" as one of the five objectives as acceptable to all regardless of nationality. Bloom (1956, p. 165) notes: "In one sense, all learning is creative; the individual has acquired an understanding or some reorganization of experience which is novel for him. The novelty for him is what makes the experience "creative".

The extent to which this objective of creative endeavor can be translated into classroom practice is uncertain. More must be known about the way students of various levels approach problems if we are to encourage creative endeavor in the school. The present study seeks to establish a model for creative behaviour in mathematics which not only incorporates the definitions of creativity, but one that is also relevant to the junior high school level. The intent was to create problem-situations calling for processes that are defined by the model, then to examine the responses and some of the procedures used by grade nine students in solving these problems.

General Creativity

Historical Views of Creativity

The concept of creativity has long puzzled man. The sudden flashes of insight which distinguish the inventor, the alchemist, or the philosopher were attributed to evil, to unnatural powers, to madness, and to God. These individuals were revered, burned, or merely shunned depending on the mood of the masses and the times. With the advent of the machine age, the concept of supernatural power fell into ill repute, and creative talent became recognized as a domain of study within the realms of psychology and psychiatry.

Traditionally the schools, being mainly concerned with the transmission of established knowledge, have left the Galileo, the Hindemith, and the Einstein, to grow as well as he would within or despite the system. As our world continues to become an environment that changes more and more rapidly, noted people in the fields of psychology and education as well as those in various other disciplines have impressed upon the world the need for an individual who has not only been exposed to the knowledge of the past, but one who is capable of using this knowledge in a new and useful (utilitarian or aesthetically gratifying) way. (Toynbee, 1964; Moustekas, 1967) The need of the world for such an individual, and the individual's own need for a development of his own creative worth, whether it is small or large, in order to retain his sanity in a world of automation, has resulted in a new demand on the school.[1] The schools are now asked not only to transmit to the child historical facts, discovered principles, and established evaluations, but to develop the means with which these basic foundations can be used to establish new footholds and worthwhile pursuits for the coming generation.

This emphasis encompasses the fact that, although there is a practical reason for our society's interest in developing creative potential, an equally important facet may be to focus on the individual and his inner need.

---

[1] i) Taylor Pearce (1971), Bloom (1956) and I. E. A. Husen (1967), include in their objectives for the school, the category of inventiveness: The ability to reason creatively in mathematics.

ii) Guilford (1959) suggests that creativity is related to learning. A look behind some of the ideas in a creative solution may help us to understand the processes by which we learn.

Underlying this focus is the assumption that every individual has creative potential. With additional recreational time available to many individuals, with the emphasis on the individual well-being, and mental health, it may be important to encourage each individual to develop his mental potential as much as possible. Barron (1968) speaks of developing an individual ability to reflect in the creation of one's own personality.

> A person may be said to be most elegant, and most healthy, when his awareness includes the broadest possible aspects of human experience, and the deepest possible comprehension of them, while at the same time he is most simple and direct in his feelings, thoughts, and actions.
> (Barron, 1968, p. 4)

Barron sums up the points discussed above in the following paragraph as he discusses his rationale for his interest in creativity.

> Governments became interested because the sheer physical power and, by a very short step, political power that comes from inventiveness has suddenly become so manifest; commerce is newly interested because the increase in goods, services, and profits is most evidently dependent on new ideas; religion is interested because old meanings have been destroyed and new ones call to be created; the individual is interested because to create is to be more fully and more freely oneself. Perhaps at no other time in all of human history has there been such general recognition that to be creative in one's own everyday activity is a positive good.
> (Barron, 1968, p. 7)

Some Diverse Approaches to Creativity

The creative act has been defined and redefined for many years. A collection of autobiographical sketches edited by Ghiselin recounts the creative experiences felt by a number of famous contributers to our society. Many of these creative impulses were discussed in terms of the classic chronological stages proposed by Wallas--preparation, incubation, illumination, and verification. (Ghiselin, 1952; Hallman, 1967)

Creativity has also been defined in terms of character traits such as dominance, the ability to tolerate ambiguity, openness to experience, spontaneity, a desire to remain outside a group learning situation. It is possessing an inner sense of values thereby being independent of standards and judgements outside of the subject. (McKinnon, 1962) It is being intuitive and free to follow the non-rational impulses.

The creative male is suggested to be more feminine, more sensitive, more intuitive and to have been closer to his mother as a youngster than the non-creative male; the creative female is said to have more of the masculine traits of independence and dominance, to have related much more to her father than her non-creative counterpart. (Roe, 1964; Maslow, 1958; MacKinnon, 1962)

The strength and power behind the creative impulse has been claimed a resultant of repression of deviant sexual behavior or desires present in the developing youngster. (Hallman, 1967)

Creativeness has been looked upon as insanity, as witchcraft, as some type of deviation from the established norm. Contrary to this view, the creative process is now being accepted as a form of psychotherapy, an acceptable way of reducing tension, a protective measure against the stresses that develop into mental illness. (Barron, 1968; Maslow, 1962)

Creative thinking has been defined in terms of types of mental operations, of "levels of psyche". Two types of thinking usually reported in any account of the creative happening are the unconscious or involuntary, and the conscious, rational controlled thought. (Ghiselin, 1952) Poincaré's account of his creative experience portrays incubation and illumination as involuntary, pressing, and having a surging power over which the individual has little or no control. The rational and controlled processes are reflected in the hard work of careful preparation and in the methodical, careful aspect of verification and communication and the elaboration and testing aspects of creative action. (Poincaré)

These descriptions specify only some of the conditions which may represent a creative act or person. Hallman (1967) claims the evidence concerning creativity clusters around five methodological approaches to the problem of defining creativity. The five approaches are listed below and the references indicated for each are those included by Hallman.

i) Identification of personality traits. (Fromm, Maslow, Barron)

ii) Identification of a series of chronological stages. (Poincaré, Ghiselin)

iii) Identification of layers of psychological systems. (Freud, Murray, Taylor)

iv) Identification of types of thinking. (Spearman, Bruner)

v) Review of personal reports. (Ghiselin, Nietzsche)

Hallman's purpose was to organize this evidence into a structure that would "reflect the necessary and sufficient conditions of creativity". (p. 18) These criteria describe the creative act in terms of five major components: the act, the object, the process, the person and the environment.

i) The act is identified by the condition of connectedness. This criterion means the fusion of elements into new structures with new relationships; it means the production of new connections between ideas.

ii) The object is identified by the condition of originality or more specifically by the qualities of "newness", unpredictability, uniqueness, or surprise which describe the invention.

iii) The process is identified by the condition of irrationality. This criterion focuses on the essential activity which occurs in the unconscious as well as the description of the mental functions which give rise to the creative act.

iv) The person is identified by the condition of self-actualization. This criterion identifies those individuals that are motivated towards personal growth, toward self-expression, towards a goal which is above mundane and material gratification.

v) The environment of the individual, both personal and social, is identified by the condition of openness. This criterion refers to the personality traits (tolerance of ambiguity, spontaneity), as well as to physical environment.

These numerous ways of looking at creativity spell out the complexity of the attribute, and indicate that any specific definition or study necessarily fragments the creative act in order to focus on some specific view.

Since the present study is concerned with a description of the mental functions by which the student achieves solutions to problems, the process view of creativity was emphasized. The processes examined during the study are not all described by the word irrational but the study seeks to establish as best as possible some of the conscious and unconscious mental processes instrumental in creative solutions to mathematical problems. These processes are defined in detail in Chapter III.

The objects of the creative process are also identified by the present study since the responses of the individuals were scored and examined for originality in terms of the procedures discussed in Chapter IV.

THE PROBLEM

Statement of the Problem

The purpose of this study is to examine creative thinking by researching the processes by which junior high students solve problems in geometry. The study proceeded in three phases.

Phase One

i) A definition for creativity in mathematics, which is appropriate to mathematical study at the junior high school level, yet is valid beyond this level, was established logically and by reference to established literature. Creativity is described by four processes to be defined in Chapter III.

ii) Specific characteristics and guidelines were established by which a set of problem-situations that measure the defined processes may be constructed.

iii) Problem-situations which require the individual to use the defined processes were constructed.

Phase Two

i) The constructed problem-situations were administered to a sample of forty-two grade nine school students. Written responses, as well as oral elaborations on the procedures by which students arrived at the written responses were collected.

ii) An appropriate scoring scheme for the problem-situations was established.

iii) The results were analyzed by correlational comparisons and by image analysis in an attempt to validate the model.
The adequacy of the problem-situations in terms of the guidelines established under the definition was discussed with reference to the statistical results of this analysis. Indications of the appropriateness of the established model in reflecting the problem-solving processes of junior high students were also considered in terms of the responses given to the problems.

Phase Three

i) The specific student answers and their additional comments taped during the administration of the problems were examined. The oral elaborations on the written responses further described the processes hypothesized for the model.

Definitions

The pertinent definitions are stated briefly. The processes of creativity and the scoring terms will be elaborated in Chapters III and IV, respectively.

Convergent Production. Convergent production is the generation of information from given information where emphasis is on an unique or most appropriate answer. The problem demanding convergent production is typically highly structured and the criteria are specific.

Convergent Process. A convergent process is a way of responding which emphasizes a most appropriate answer and results in convergent production.

Divergent Production. Divergent production is the generation of information from given information where emphasis is on quantity and variety of output from one stimuli. (Guilford, 1967, p. 213) The problem calling for divergent production is loose and broad in its requirements.

Divergent Process. A divergent process is a way of responding which emphasizes the generation of many appropriate answers to a given question.

Creativity. Creativity is defined as four processes—conjecturing, sensitivity, redefinition, and verification. (The definition is developed in Chapter III of the report.)

Conjecturing. Conjecturing is a divergent process which refers to the generation of hypotheses, of relationships, in response to a given source of data. The individual may conjecture by making naive guesses or by making deductive guesses.

Sensitivity. Sensitivity is a divergent process which includes the ability to perceive deficiencies and errors, shortcomings or inadequacies in a given situation, and the ability to see possibilities in a given situation; possibilities that lead to further questions.

Redefinition. Redefinition is a convergent process which includes the reassociation and the recombination of previously unassociated elements of knowledge to result in new combinations, and the discarding of previously adequate approaches in order to facilitate the perception of and the solution to a problem.

Verification. Verification is a convergent process which refers to the justification of a statement or relationship in four ways: by testing with specific examples, by establishing a rationale of assumptions, by producing suggestions by which a statement may be tested, or by formulating a deductive proof.

Fluency. Fluency refers to the quantity of response given by an individual to a given situation. The fluency score for the divergent-production problems in the study is given by the number of appropriate responses.

Variety. Variety refers to the number of different types of responses given by an individual to a given situation. The variety score for the divergent-production problems in the study is given by the number of different categories of responses.

Novelty. Novelty refers to the uniqueness, the uncommonness of a response. The novelty score for the divergent-production problems in the study is given by a rank assigned to the statistical frequency of a response. The more frequently a response is given, the lower the novelty score.

Significance of the Study

Describing situations which reflect processes involved in creative thought has some relevance and implication for the teaching of mathematics in the classroom. The assumption underlying this study is that creative thought can be inhibited or encouraged, the quality of creative thought can be improved and can be taught for in the classroom.

The scope of creative thinking is very broad and has been considered in many ways, some of which have been indicated in the above discussion. Further understanding of the ways in which students perceive relationships and concepts in mathematics, of the approaches students use when facing a new problem, of the incomplete and inappropriate associations present in the student's mind can add to the classroom teacher's awareness in the area of mathematics and in the area of thinking. This awareness may be helpful in structuring some of the learning tasks in mathematics so as to allow the children to explore and to expand on the incomplete concepts that they possess. If certain problem-solving strategies can be identified as ones which children use, teachers may incorporate these into their discussions, or their presentation. It may be very important to let a student choose his direction to solving a problem, then to discuss or indicate to the student at which points this solution is weak or does not meet necessary criteria.

The ordinary classroom teacher does not have much time to explore an individual student's train of thought as he learns mathematics, however if creative thinking can be presented in terms that are understandable and related to material and happenings which occur every day in the classroom, the teacher may devote more time to this aspect of the curriculum.

The present study has focused on creativity for a specific content area. This focus should be simpler to apply to the ordinary classroom situation than a more general view of creativity.

The main emphasis in the study is on the processes behind creative problem-solving. These processes are units that are larger than those used in psychological studies such as Guilford's Structure of the Intellect, but because of this, may be more easily identifiable in the context of the everyday classroom.

Delimitations

The study was delimited to forty-two grade nine students in four Edmonton Public Schools. The constructed problems were limited to those that require for solution only those concepts and skills included in the junior high school program for geometry in Alberta schools. The following is a list of the ideas that were considered necessary knowledge for the study. (Department of Education, 1971; Edmonton Public School Board, 1971)

i) An understanding of points, lines, planes, space.

ii) An understanding of angles.

- to know the relationships of vertically opposite angles and supplementary angles.

iii) An understanding of polygons.

- to know the characteristics of a square, triangle, isosceles triangle, equilateral triangle, rectangle, parallelogram, circle, the right triangle.
- to know that the sum of the measures of the interior angles of the triangle is 180°.
- to know that the measure of the angle described by a circle is 360°.
- to know the relationship between the length of the hypotenuse of a right triangle and the length of the other two sides.

iv) An understanding of area and perimeter.

- to know how to calculate the measure of area for triangles, rectangles, circles, and parallelograms.
- to know how to calculate the measure for perimeter for these figures.

v) An understanding of percentage.

- to know how to calculate questions like 3% of 16, and 6 is _____% of 18.

Limitations

The study is exploratory in nature, therefore specific procedures for scoring were determined from the responses, both written and oral, given to the problems. One other individual scored the responses to the conjecturing situations for a small sample of the students taking part in the experiment. This provides a serious limitation on the reliability of the scores used in the statistical analysis and for this reason the statistical analysis was used mainly to quantify the description of the processes rather than to validate the measuring instrument. Since there was only one observer in the study, the reliability of the task administration is also not known.

The validity of the instrument for measuring creativity is limited by the imposition of a time limit on the students' solving of the problems as well as by the interpretation of responses by the single observer and interviewer. This interpretation was particularly important since a criterion of appropriateness was applied to the responses for scoring. Although the researcher attempted to conduct the study without any pre-experimental bias, each child had different questions and a different attitude to the study. An attempt was made to be as supportive as possible. Since this attention was personal to each child some bias may have resulted in the way each problem was presented from one child to the next.

The students were chosen for the study on the basis of convenience for the schools, resulting in a sample that was not random. This difficulty combined with that of small numbers in the sample, the inductive approach used to establish the scoring schemes, and the decision to concentrate on an explorative and descriptive examination of student responses limit the extent to which the results of the study may be generalized. The results only indicate the nature and the description of the processes that may be used by students in solving problems; they do not predict their occurrence in the general population. As such they can only be used as guidelines for further exploration.

An Outline of the Report

The study is organized around eight chapters. The literature most relevant to the interpretation of creativity and for the development of a model for mathematical creativity in terms of problem solving is presented in Chapter II.

The model is presented in detail in Chapter III. The processes hypothesized to describe creative problem-solving are defined, characteristics of situations which reflect these processes are discussed, and outlined, and two problem situations designed to require each process are presented.

In Chapter IV, the design for phases two and three of the study as well as a discussion of the statistical procedures used is presented.

In Chapter V, the scoring procedures are established with reference to literature and to the specific responses given to the constructed problem-situations.

In Chapter VI, the results to the correlational and image analysis are presented and used to answer the following questions which represent the major hypotheses for the study.

i) Do the scores for the problem-situations separate into four distinct image factors which can be described as the four processes stipulated by the model established in Chapter III?

ii) Are the scores for the two problems representing each process correlated? More generally, does the first conjecturing problem reflect the same process as the second conjecturing process?

iii) Are the scores for the problems of conjecturing, sensitivity, redefinition, and verifying significantly intercorrelated?

iv) Are the scores for the experimental problems significantly correlated with the scores for tests of ability, achievement, and creativity?

This chapter seeks to establish the degree to which the results from scores on the constructed problem situations can be described by the model. The weaknesses and the appropriateness of each problem are also discussed in terms of the results.

In Chapter VII, some of the individual responses and oral comments provided by the students are presented in an attempt to describe the process of conjecturing as used by the students. This chapter attempts to find some levels of response and some categorization of approaches that describe individuals who produced creative responses.

The summary and suggestions for further research are presented in Chapter VIII.

CHAPTER II

A REVIEW OF THE LITERATURE

The discussion in the previous chapter outlined the many approaches to a definition of creativity. In this section, five main approaches will be reported in greater detail. The fifth approach, the process view, will be emphasized and chosen for the present study. The discussion of especially this section will point the way for establishing the model to be used in this discussion.

Reputable researchers, such as Getzels and Jackson (1962) and Cattell (1971) have agreed that the creative individual must have a certain level of ability as defined by intelligence studies, but that he must also possess some other traits as well. These may be environmental, personality, or some special mental abilities. The focus depends upon the writer discussing creative endeavour.

It is not the purpose of this study to determine the effect of intelligence on creativity, nor to argue the relative effects of environment or personality, but to choose a definition of creativity that will enable the author to describe responses of junior high students to mathematical problems, thereby achieving some information as to how to best plan activities that will enable each student to develop the creative potential that he possesses.

The following literature report will present five views of creativity in order that the reader will have some understanding of why the process view was thought to be most acceptable. This view will then be expanded to include problem solving. The final working definition will be a result of the ideas discussed in the connection between problem solving and creativity.

FIVE VIEWS ON THE NATURE OF CREATIVITY

Chronological Stages

Creativity was first described in historical accounts by notable individuals in science and literature. One of the accounts most pertinent to this study was made by Poincaré, who pinpoints four distinct stages in his account of a mathematical discovery.

> Most striking at first is this appearance of sudden illumination, a manifest sign of long, unconscious prior work. The role of this unconscious work in mathematical invention appears to me incontestable, and traces of it would be found in other cases where it is less evident. . .
> There is another remark to be made about the conditions of this unconscious work: it is possible, and of a certainty it is only fruitful, if it is on the one hand preceded and on the other hand followed by a period of conscious work. These sudden inspirations . . . never happen except after some days of voluntary effort which has appeared absolutely fruitless and whence nothing good seems to have come. . . The need for the second period of conscious work, after the inspiration, is still easier to understand. It is necessary to put in shape the results of this inspiration, to deduce from them the immediate consequences, to arrange them, to word the demonstrations, but above all is verification necessary.
>
> (Poincaré, p. 38)

Wallas (1926) described these stages as preparation, incubation, illumination and verification; and cites in his defence of these stages accounts by such notables as Hemholtz, Hobbes, Wallace, Darwin, and James. Several other individuals (Taylor-Pearce, 1971, p. 15) have outlined variations of the four steps, however the basic ideas behind the stages are very similar.

During the first stage, preparation, information is assimilated, explored, outlined, and organized. This work enables the individual to be very familiar with the ideas behind the problem and enables him to manipulate the ideas easily in his mind.

At this point, a period of incubation is necessary. Poincaré refers to this as the unconscious work of the individual, a period when the individual stops his conscious concentration on the problem at hand and allows the unconscious to rearrange the information into new combinations and new ideas. Dewey explains that when the consciousness has relaxed its strain, the material often rearranges itself, and ideas, that were confused become ordered. At this point a solution often occurs.

Illumination is this insight into the solution, whether it is sudden and total, as described by Poincaré, or whether it appears in successive stages of inspiration and clarification. (Ghiselin, 1952, p. 30) In the latter case, the two stages of illumination and verification blend into one.

During the verification stage, the insights of the previous stage are verified, checked against realities, and ideas are elaborated or extended.

> After that excitement is dissipated, its intrinsic value is its only relevant one even to himself. He must find out if it will serve to organize experience in a fresh and full and useful way. To that end he tests it critically.
>
> (Ghiselin, p. 30)

These four stages defining the chronological description of the creative act have influenced the development of problem solving models (Dewey, 1910; Guilford, 1967). This relationship will be elaborated upon later in the chapter.

In addition, awareness of the four stages establishes some idea about the conditions necessary for creative problem solving. An attempt must be made to ensure that subjects asked to respond creatively have had the opportunity for preparation and if possible have had time to deal with the content at some leisure.

One other noteworthy feature of this view of creativity is the equal consideration of verification, especially in the case when solution and verification interact in alternate sequence. The realization that each specific insight may be dependent upon the verification or rejection of a previous idea, makes the inclusion of this stage important for any model of creativity.

Personality and Environmental Traits

The second approach to defining creativity depends upon the description of creative individuals in terms of personal and sociological conditions. Included are characteristics of the individuals' external environment as well as inner resources and traits. Hallman (1967), Taylor (1962), Jackson and Messick (1965) all consider traits such as sensitivity, (the ability to be aware of deficiencies, to be aware of the obvious, of the external environment, to be puzzled), tolerance of ambiguity, and spontaneity to be important to the creative individual.

The empirical studies of character and creativity have come from two sources, first the study of biographical data of scientists accepted as creative by time, and second, the description of contemporary researchers, identified as creative by their peers.

Roe (1952), and Cattell (1959) reported eminent scientists from biography to be introverted, dominant, and self-sufficient. Cattell and Drevdahl (1955), extending the research to physicists, biologists, and psychologists deemed creative by their peers, found that i) the scientific researcher is more introverted, more dominant, more inhibited, and more emotionally sensitive than the average individual; ii) the scientific researcher is more introverted, less stable, more radical and more self-sufficient than another professional of equal intelligence and education; iii) the scientific researcher is very similar to his artistic or literary counterpart, except that he is somewhat more stable and less tense. Roe also found that the social scientist differed from the physical and biological scientist, in that he tended to be an extroverted individual much more concerned with social interaction than his physical or biological peer.

McKinnon (1962) found that his sample of creative architects shared these traits and also were very marked by their receptiveness and openness to richness and complexity of stimuli as compared to other professionals and to less creative architects. Jackson and Messick (1965) explain that the reflective individual has the ability to condense the complex into a simple unified work which has about it an "intensity and concentration requiring continued contemplation." (p. 321)

Barron (1955), Taylor (1957) found army personnel and college students that were deemed creative to be more self-sufficient, more able to stand stress, more independent of values external to themselves than their less creative counterparts.

Personality studies, like the above have attempted to describe some of the character traits that distinguish the creative from the less creative. The danger lies in predicting the existence of creative potential from the character traits. An introverted, non-conforming cynic is not necessarily a creative individual.

A new and more fruitful direction for describing the character and the environment of the creative lies in the establishment of an environment that will encourage the development of creative potential in every individual. Educators, such as Smith, Torrance, Osborne, and Parnes have established some guidelines for programs that have as their objective the increase of creative potential, but at present there is disagreement as to what conditions produce what effects. Stratton and Brown (1972) found that creativity programs which combined judgement and production training resulted in an increase of number of responses and quality of responses as measured by the Plot Titles Problem, when compared to separate training or no training. Clark (1968) found that students in Grade VIII science taught by traditional teacher-centered methods increased their scores on fluency and originality (Torrance) more significantly than did those students taught according to Suchman's Inquiry Training

Method. Studies such as these vary so greatly as to standard procedures, means of checking achievement and creative ability that it is very difficult to draw any definite conclusions at this time.

Psychic Levels

Historically creativity was linked with the subconscious and the unknown energies present in the human being, and contrasted with the conscious energies which we can willfully control. These mystic powers were linked with the psyche, with the powerful repressed drives of sex and with madness. Recently, however there has been a break from the mystical aspect in an attempt to organize the levels of creativity into a classification useful for description or development. Irving Taylor (1959) has suggested that creative behavior is possible at five levels. The first level is an expressive creativity, spontaneous, but one in which the quality of the product is relatively unimportant. A second level is that of productive creativity. The emphasis is on an improved technique and controlled free play, however is still very likely to produce products of inferior quality. Inventive creativity involves a perception of new and unusual relationships between previously separated parts; innovative creativity, undertaken by very few, results in modification of basic foundations, and concepts in a field of study. The top level, the emergentative level, results in a most abstract and fundamental form of a product.

The above approach may initiate several questions in the minds of teachers. Which levels of creativity can be developed in the classroom? (The experimenter would suggest that some part of the junior high population can work at the third level.) Is the development and the encouragement of even the lowest level of creativity important enough to warrant classroom emphasis? How do we distinguish between creativeness at this bottom level from permissiveness and lack of standard?

In the present study, this latter discernment will be established by the questions constructed. The responses may not be original in the absolute sense, but many of them may be original to the student. Students will certainly be aware that the answers will have to be appropriate in the mathematical sense.

The Products

The previous categorization infringed upon this one in that the product was used in defining some of the categories. Historically the product has been the final determining factor by which an individual is deemed creative, but as with other criteria, there are difficulties. Should the result be new to the world or should it be new to the discoverer? Kneller (1965, p. 3), Hadamard (1945, p. 104), Taylor-Pearce (1971) support the latter. However Kneller (1965) notes a distinction of degree between the creativity exhibited by the discoverer and that of the rediscoverer. (p. 3)

Guilford has used the concept of unusualness or novelty in comparison to other people. A response to one of his problems is novel if it isn't reproduced by a numerically significant number in his sample. Taylor-Pearce has introduced the importance of appropriateness. A product may be statistically rare, but may be a totally irrelevant or a nonsensical idea. The specific application of these ideas will be discussed later in connection with the scoring scheme. (Chapter V)

Messick and Jackson (1965) have also used the ideas of appropriateness and usefulness in their classification for evaluating responses and products. (p. 312-313) A class of potentially creative products are limited by these two criteria. Further distinction in quality and level of responses are judged in terms of transformation and condensation. Transformation (p. 315) is the degree to which the products involved have overcome conventional constraints. The constraints themselves are also evaluated; the nature and strength of these constraints determine the creative value of the product. Condensation (p. 320-323) includes the complexity and the "summary power" of the response. A creative product has a multiplicity of interpretation, yet is unified by a basic simplicity that binds together the complex elements.

The above ideas have further implication for building a task which will elicit a creative response. Such a task should allow an individual to work at many levels of interpretation, and to be able to unify several interpretations, several ideas into a unified structure. Transformations of the given data should also be possible for the student.

The Processes

Webster defines process as the succession of related changes by which one thing gradually becomes something else, a particular method of doing something, of producing something. The process view of creativity seeks to explain the creative act in terms of the means (the processes) by which the final act is produced. Guilford is one of the most well known writers who incorporates process into his definition of creativity. This writer has described human intelligence in terms of three dimensions--Products, Content, and Operations. The five operations (or processes) include cognition, memory, convergent thinking, divergent thinking and evaluation. Guilford created a large battery of tests and by factor analysis related these test results to the hypothesized categories defined by the three dimensions. From his findings he defines creative thought largely as divergent production. He found however that factors of redefinition were convergent production; sensitivity to problems was evaluation. (1967) (He later recategorized it into the cognitive area).

Table I shows the convergent and divergent rows as presented in the semantic slab of the Structure of the Intellect cube. These two processes are focussed because they underline the scoring procedures used in the present study.

Guilford's study has been criticised from two points of view. The first questions the technique of establishing an a priori model and the using of factor analysis procedures to confirm the

TABLE 1

GUILFORD'S STRUCTURE OF INTELLECT

A SECTION OF THE SEMANTIC SLAB OF GUILFORD'S MODEL - CONVERGENT PRODUCTION AND DIVERGENT PRODUCTION

	OPERATIONS	
PRODUCTS	CONVERGENT PRODUCTION	DIVERGENT PRODUCTION
Units	concept meaning, subsuming several given ideas for new idea	ideational fluency little or no restriction
Classes	organizing ideas into meaningful classes	spontaneous flexibility--number of varied responses
Relations	to produce an idea related to another in a specific way	associational fluency to produce manipulations having an idea in common
Systems	to generate a system from several ideas, order a sequence of steps to complete a task	expressional fluency, many sets of ideas
Transformation	redefining change in interpretation or emphasis	originality to produce effective surprise transformations unusual ideas
Implications	to deduce a statement of results from a set of ideas	elaboration of an idea, deductions or inferences

*Units - relatively segregated or circumscribed items of information, a "thing", something of character.

Classes - sets of items of information, the emphasis is on attributes, properties.

Relations - recognized connection between items of information based upon applicable variables.

Transformations - changes of various kinds of existing known information in attribute, meaning, role or use.

Implications - expectancies, anticipations, predictions, of one item to another.

Cognition - awareness, immediate recognition, comprehension, understanding.

Convergent Production - logical deductions compelling inferences, an unique answer.

model. Cattell (1971, p. 412) states "Indeed, we need constantly to be reminded that when tentatively we conceptualize the behavior of rigidity, fluency, and flexibility as expressions of cognitive performances, we are actually in a complex and insufficiently analyzed field in which much of the variance probably will turn out to be due to personality and temperament factors. . ."

Second, Guilford's tests have been considered irrelevant measures of creativity. For instance, an individual who can find many unusual uses for a brick may or may not be creative in reality. This definition of creativity in terms of performance on a constructed test of this type results in creativity as a projection of the author's personal bias. (Cattell, 1971, p. 408)

These criticisms do not seriously affect the present experiment. The model for creativity that is to be defined does not depend upon Guilford's model or his tests. The scoring scheme, however, does use the philosophy that the number of responses, the number of different responses and the unusualness of response indicate the creativity of an individual in connection with a divergent production situation. These ideas are only used to describe more specifically the responses given by students to specified situations. The situations to be chosen are problems which represent the field and structure of mathematics. The concepts of fluency, flexibility and originality have been modified to include a consideration of the appropriateness of the response. The foundations behind the scoring scheme will be considered further in Chapter V.

Recently, other researchers in education have taken the process view of creative thought. Evans (1964) discusses intuitive thought as the common element in mathematical discovery learning, in problem solving and in transfer; then proceeds to measure the process. Smith (1967) describes creativity as "a mental process that involves tapping one's experiences and rearranging them into something new." (p. 2) Torrance (1963) and Simon (1967) describe creativity in terms of problem-solving processes.

Since the researcher was concerned with describing students' thinking as they proceeded through their mathematical problems and thereby distinguishing some of the features which resulted in effective and new solutions, the process view seemed most appropriate. The next section justifies the description of creative process in connection with problem-solving process, and leads to the development of the definition to be used in this study.

CREATIVE THOUGHT AS PROBLEM SOLVING

Creativity, up to this point, has been discussed in general terms; has been referred to as a general trait. There is some question, however, as to whether creativity is a general intellectual factor or whether it is specific to some discipline or to some situation. Brandewin (1960, p. 67) states that the consensus is that the creative process is the same for all subject matter, but that it is quite likely that there are patterns of intellectual and personality make-up which exert an influence on creativity in a given field of work. Spieth (1963) compared the intelligence of students judged creative in literary, scientific, artistic areas or in more than three areas, by teachers who had had them over a period of ten years. He concluded that sufficient intellectual differences existed between these four groups, thus making it unwise to combine fields together when correlating creativity and intelligence.

Certainly, if a measure or description of creativity is to be most appropriate to a certain area, it will be best described in terms of that discipline. The researcher's prime interest is in describing thinking as it occurs in the mathematics classroom in response to mathematical content.

Does much of what has been already said about creativity apply to mathematics and creative thought in this field? Mathematics has been described in terms of two aspects. Like all sciences, it is intuitive. Hypotheses are generated, conjectures are made, approaches to problems are based on a "hunch". But mathematics diverges from this scientific type of exploration in its deductive form. Because it is based on man-made assumptions and not on the physical world, mathematics is capable of an absolute truth with respect to the scientific system. Lovell (1966) makes the following statement about the study of mathematics:

> It can be regarded as a totality of deductive "theories", all of which are grounded in pure logic, or as an autonomous activity of the individual, the ultimate source of which is the primordial faculty of intuition. However, the former viewpoint does not appear to be in favor in many quarters since the metamathematical theorem of Godel implies that formal logic is incapable of ever containing the whole of intuitive mathematics. Although formal logic is indispensable to mathematics it does not appear to be able to provide the ultimate criterion of the validity of mathematical assertions.
> (p. 209)

Polya states that mathematics appears in two guises—as the rigorous systematic science of Euclid, and as the mathematics in its making, an experimental and inductive science. (Polya, 1946, p. vii) Courant and Robbins (1941, p. xvii) point out that constructive invention and intuition are at the heart of mathematical achievement and provide the driving force, but that the goal is a neat logical deductive system which gives a better understanding of mathematical fact and relationship.

These statements indicate that mathematics has the two aspects thus far discussed for most creative work--the producing, generating phase, and the verifying, directive phase. These two aspects will be included in the definition used in this study.

The intuitive and the deductive aspects are included in definitions of creative thinking that relate to problem-solving. Torrance, who has been involved in much of the research on creative thought at the school level, defines creative thinking as the "process of sensing difficulties, problems, gaps in information, missing elements; making guesses or formulating hypotheses about these deficiencies; testing these guesses and possibly revising them; and finally in communicating these results". (1965, 1971. . .) These processes are often used in definitions of problem-solving. Dewey (1910, p. 72), for example, proposed the following steps for problem-solving: a difficulty is felt, the difficulty is located and refined, possible solutions are suggested, consequences are considered, and then a solution is accepted. Polya (1946) states that the following steps be followed in solving a problem:

understanding a problem, devising the plan, carrying out the plan, looking back and verifying the solution in terms of the original problem.

Torrance maintains that the similarity to problem solving is valid. He refers to Newell, Shaw and Simon's view that problem solving is creative to the extent that it meets one or more of the following criteria:

i) novelty or value, social or personal, possessed by the product.

ii) unconventionality of the thinking, requiring modification or rejection of a previously held idea (redefinition).

iii) motivation or persistance level is high.

iv) vagueness of the problem, requiring formulation of the problem. (Torrance, 1967)

Simon makes a more direct statement:

> My first hypothesis is that the creative processes, the processes a person uses when he's doing creative thinking, are indistinguishable from ordinary problem solving processes. What distinguishes the creative thinker from any person who is solving problems is only the distinctiveness of the product: that his solution is a novel, valuable unconventional result.
> (Simon, 1967)

Guilford agrees with this point of view and states that "There is something creative about all genuine problem solving and creative production is typically carried out as a means to the end of solving some problem." (Guilford, 1967, p. 314) In summary is presented Guilford's illustration comparing the classical definitions for problem-solving per Dewey and Wallas to a model of creative behaviour by Rossman. (Table 2)

TABLE 2

STEPS IN THE SOLUTION OF A PROBLEM, IN CREATIVE PRODUCTION, AND IN INVENTION, AS SEEN BY DEWEY, WALLAS, AND ROSSMAN, SHOWING SIMILARITIES AND DIFFERENCES

DEWEY	WALLAS	ROSSMAN
Difficulty felt		Need or difficulty observed
Difficulty located and defined		Problem formulated
	Preparation (information gathered)	Available information surveyed
	Incubation (unconscious work going on)	
Possible solutions suggested	Illumination (solutions emerge)	Solutions formulated
Consequences considered	Verification (solutions tested and elaborated)	Solutions critically examined
		New ideas formulated
Solution accepted		New ideas tested and accepted

The experimenter has presented the above literature in order to defend the premise that creativity can be validly defined in terms of problem-solving processes. It is necessary to establish what some of these processes may be and to describe them specifically enough so that mathematical problems requiring each of these processes may be constructed. The following section will present three theories describing mathematical problem-solving. The problem solving referred to by each of these individuals is not the solving of the familiar problems by familiar recipes, in routine work, but the solving of new problems, problems requiring some degree of independence, judgement, and originality. This type of problem solving may be considered cognate with problem-solving. (Cattell, 1971, p. 428)

Guilford

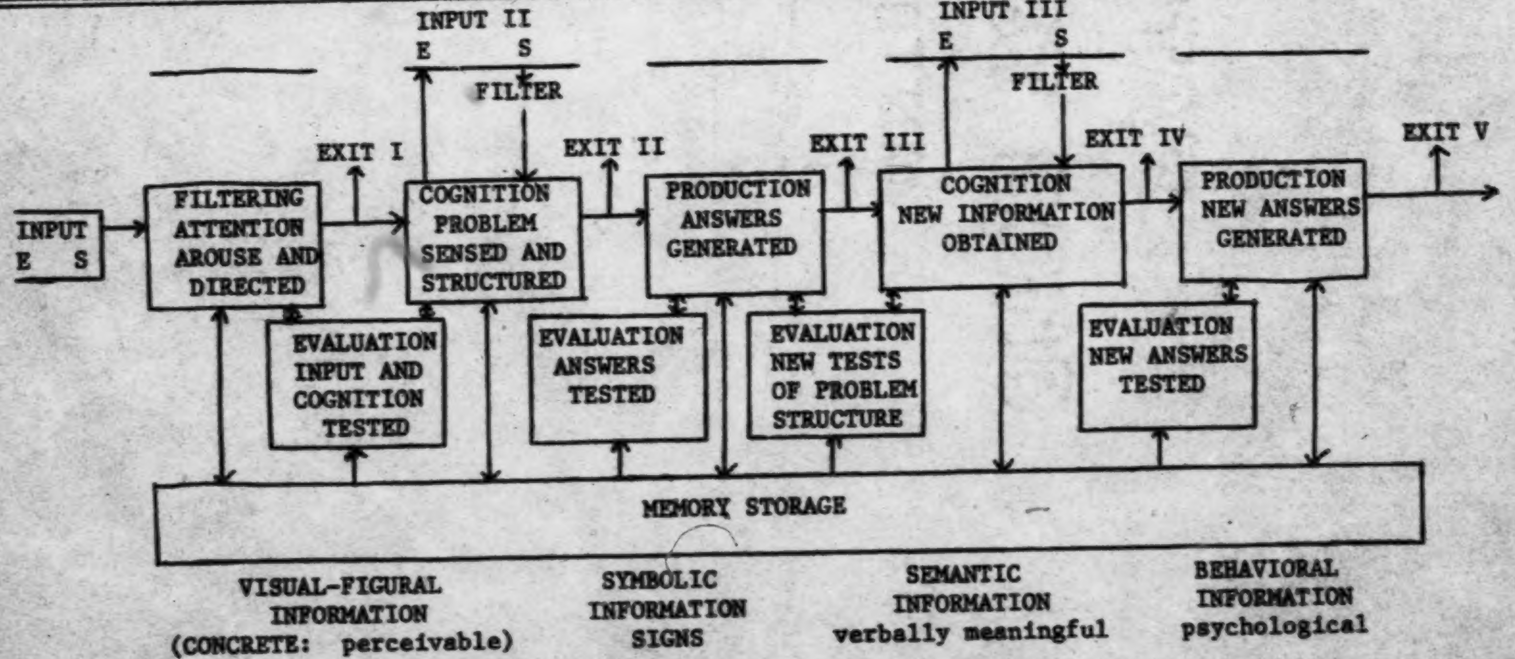
As stated before, Guilford favors a problem-solving approach to creative thought. He has incorporated his divergent and convergent production abilities into a model which illustrates their possible interaction. (Table 3)

Note the general problem solving categories of problem sensed, solutions generated, solutions tested, categories which reappear in a continuous cycle. Notice also that evaluation is present throughout the activity.

It is questionable as to whether the evaluation at the production stage is the same as that occuring at the filtering stage

TABLE 3

AN OPERATIONAL MODEL FOR PROBLEM SOLVING IN GENERAL, BASED UPON CONCEPTS PROVIDED BY THE STRUCTURE-OF-INTELLECT MODEL*


INPUT
E S
FILTERING ATTENTION AROUSE AND DIRECTED
EXIT I
EVALUATION INPUT AND COGNITION TESTED
INPUT II
E S
FILTER
COGNITION PROBLEM SENSED AND STRUCTURED
EXIT II
EVALUATION ANSWERS TESTED
PRODUCTION ANSWERS GENERATED
EXIT III
EVALUATION NEW TESTS OF PROBLEM STRUCTURE
INPUT III
E S
FILTER
COGNITION NEW INFORMATION OBTAINED
EXIT IV
EVALUATION NEW ANSWERS TESTED
PRODUCTION NEW ANSWERS GENERATED
EXIT V
MEMORY STORAGE
VISUAL-FIGURAL INFORMATION (CONCRETE: perceivable)
SYMBOLIC INFORMATION SIGNS
SEMANTIC INFORMATION verbally meaningful
BEHAVIORAL INFORMATION psychological

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*Ref. Guilford, J. P., The Nature of Human Intelligence. McGraw-Hill, 1967, page 315.

or the cognition stage. Does it differ in level or just in intensity? Is there an optimum level of evaluative rigor that can be applied at each of the levels to allow for a creative response? These are questions which will be noted again during the present study.

Guilford's problem solving model includes both divergent and convergent production under the production station. He maintains that this is valid since both types of production call for the same psychological events. The difference is in the requirements imposed upon the problem. The type of response is completely specified in convergent production, but not in divergent production.

There is some question in the experimenter's mind whether a problem requiring many responses is treated the same way by a student as is the problem requiring only one response. There may be personality factors which bias an individual toward one or the other. This bias may determine the effectiveness with which a child responds to one type of question or the other.

This model also suggests that hypothesizing solutions and testing them are not independent events but that they interact and depend upon one another. It is important to remember therefore that problems may minimize relative interaction but any problem which is not artificial will require both types of thinking.

Polya

Polya, who implies creativity during his discussions but concentrates mostly on the means by which everyone can learn some basic approaches to mathematical problem solving, outlines four steps in the solving of any problem: understanding the problem, devising a plan, carrying out the plan, and looking back. These steps or some variation of these steps pervade any discussion of the scientific method.

The first phase includes an understanding and an acceptance of the problems by the solver. Questions like "What is the unknown? What is the data? What is the condition?" are important for separating and interpreting the various aspects of the data, of establishing the important variables.

Devising a plan requires the individual to "find the connection between the data and the unknown". Polya has established a number of suggestions or plans for attack that are techniques used by mathematicians and to which a novice problem solver may refer. Ideas such as looking for a similar problem or often a simpler analogous problem, working backward, generalization, and induction, are all used in various combinations to solve problems. Polya suggests that the teacher present the student with sample problems from various methods and thereby increase each student's repetoire of responses when he is confronted with a mathematical problem.

Polya suggests that from the given data, new problems may result during the solving of the given problem. He tells the student to ask such questions as "Can you solve a part of the problem? Keep only a part of the condition, drop the other part;

how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data or both if necessary, so that the new unknown and the new data are nearer to each other?" (1945)

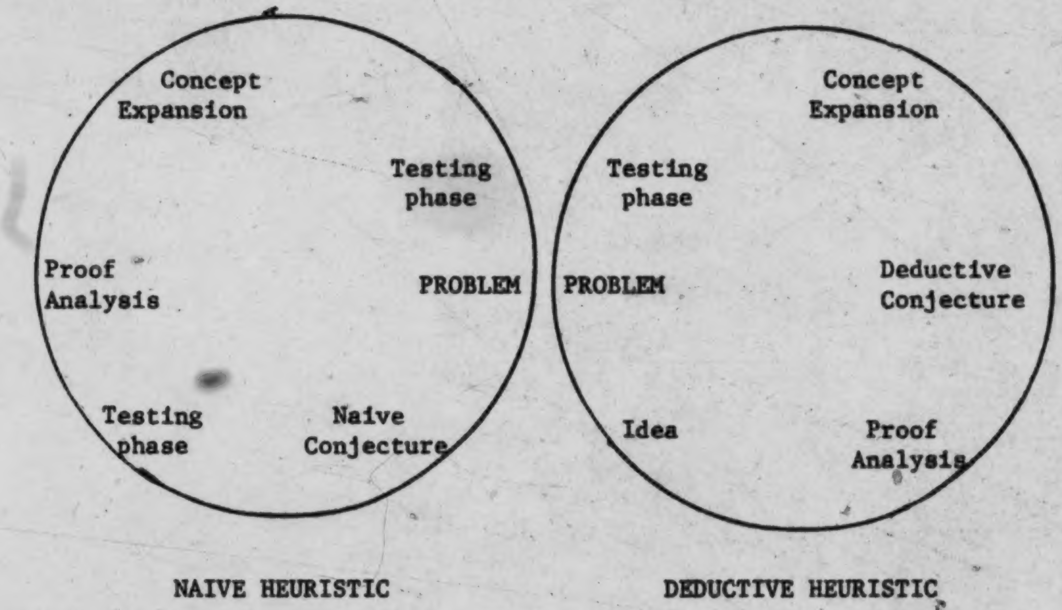
Carrying out the plan is to use the chosen of the above methods, checking each step, towards the solution of the problem.

The process of looking back is the process of checking the result for validity. The plan of attack should be checked for efficiency, the solution for correctness. At this time, the plan and the result are evaluated for their use in other problems or for new learning.

Polya's model echoes some of the ideas behind Guilford's model. Polya states that the problem must be understood; Guilford includes a cognition station, a point at which the problem is sensed and structured. These ideas relate to Torrance's first process in his definition, that of sensing difficulties. The inclusion of this process remind us that the trait of awareness has been found a characteristic of the creative. (refer to page 23, Personality and Environmental Traits) The resultant definition of creativity should then attempt to encorporate this process.

Polya also distinguishes between two types of research, the finding of and the proving of a solution. The latter process becomes the formal discussion and the verification of the statement. The former deals with the identification of the unknown, a defining of the condition which the unknown is to satisfy, an exploration of the data from which a connection is drawn to relate the unknown and the condition. According to Polya, this is the aspect that is influenced by the creative process.

Dawson (1969), discussing Polya in terms of Lakatos and Popper, defines a mathematical mode of inquiry which he symbolizes in terms of the following model: (p. 143)



As in Guilford's model there is a strong interaction between the hypothesizing and the verification stages. The testing of a conjecture in the above model is repeated continually on the basis of Popper's contention that only refutation of a conjecture is conclusive. A conjecture is only strengthened by attempts to prove it false. Dawson summarizes "Only refutation spurs the growth of knowledge for refutation demands a renewed attack on the problem-corroboration does not." (p. 135)

Dawson's discussion of Polya has drawn attention to two aspects of verification, that of testing and that of proof analysis. It becomes important to ask the extent to which junior high students will be capable of performing the latter, or to what extent they even will consider using a formal argument.

Polya also suggests specific procedures which can be used in solving a problem. His suggestions of creating analogous or simpler problems, and of induction call for a rearrangement, a transformation of elements within the problem. Some similarity in concept may be drawn between this ability to vary data and Guilford's concept of flexibility. The convergent production section of Table 1 includes i) "to generate a system from several ideas, ii) to produce an idea related to another in a specific sense, iii) to redefine, to effect a change in interpretation or emphasis". These variations are exemplified by the variation on a single hypothesis drawn up by Heinke who extended and elaborated on Polya's variation of data. In his study Heinke compiled and presented means by which conjectures are made, and organized these ideas which reflected a high school geometry classroom. (Heinke, 1953) According to Heinke, variation can occur in two places:

i) A variation is made in the data: the resultant effect is then checked in the conclusion.

ii) A variation is made in the conclusion; the resultant effect in data is analysed.

Variation occurs in several different ways, by addition to data (or conclusion), by deletion of data (or conclusion), and by substitution into the data (or conclusion). This variation may or may not necessitate a change in conclusion or data respectively.

Substitution in the data is specified as occurring through several different ways:

i) substitution of certain numerical elements to other specific numerical elements. This checks to see if the idea holds for conditions other than the given one.

ii) substitution of numerical data by non-numerical terms. Instead of a certain angle being specified in degree, information about the angle may be given in terms of isosceles triangles, congruent triangles or may be opened to include any triangle.

iii) substitution of non-numerical data by numerical data. This is examination of the general case in terms of specifics. "This is often done in initial steps of the hypothesizing process." (Heinke,

iv) substitution of generalized data in the figure of other data. This may entail a change in relationships; a change in the basic figure use. Some examples may clarify this category.

> (a) Variation--substitution involving a change in relationship eg. statement: If two angles have sides perpendicular, right side to right side, left side to left side, the angles are equal.
>
> Variant--If two angles have sides parallel, right side to right side, left side to left side, the angles are equal.
>
> (b) Variation--substitution involving a change in relationship statement. If two sides of a triangle

are equal, the angles opposite these sides are equal.

Variation--If one side of a triangle is greater than a second side, the angles opposite the first side is greater than the angle opposite the second side.

(c) Variation involving the basic figure.
Statement--The angles of a triangle add up to 180°.
Variation--The angles of a rectangle add up to 360°.

Variation may also occur through any combination of the above.

Besides the use of variation, logical transformations such as the converse, the contrapositive, or the inverse of a statement may be used to formulate a second hypotheses. The principle of duality is a variant of these and is exemplified by the following two statements:

i) Two lines determine a point.
ii) Two points determine a line.

The above suggests that another process which contributes to creative problem solving is the ability to vary and transform data. This process of transformation is not only an important ability in Guilford's Structure of the Intellect but Messick and Jackson also distinguished the creative quality of products in terms of transformation. (refer page 27) In summary, it is contended that the process of transformation is important in the description of the mathematical creative process.

A Mathematician's Model

Dr. Freedman, a mathematician and professor at the University of Alberta, suggested in a personal communication, that a mathematician proceeds in exploring a new problem in the following ways:

i) Conjecturing: The conjectures presuppose a segment of information or data, a problem or relation already posed. This information can then be varied by a conjecture which is made in one of two ways.

(a) If the hypothesis is maintained, can the conclusion be varied?

(b) If the conclusion is to remain unchanged, what variation is possible in the hypothesis?

His example is quoted:

If a function is differentiable, then it is continuous.

Possible conjectures: (a) What other conditions can determine continuity?

(b) Are these hypotheses stronger than the conditions of differentiability?

(c) If a function is to remain differentiable, what are the implications, other than continuity?

ii) Testing Hypotheses

(a) By proving conjectures--the use of the trial and error method.

(b) By constructing counter examples.

Dr. Freedman feels that it is in testing hypotheses that real ingenuity is necessary.

iii) Making Axioms

Dr. Freedman suggested that this aspect takes place very infrequently in history and thus would not be useful in a model for evaluating creativity in children.

This view of an individual actually engaged in the discipline, coincides greatly with the views presented above. Again we notice the interaction between making hypotheses, and testing them. Similarly, we again see that new ideas can be generated by means of varying and transforming the data. The systematic approach to data or to conclusion is the same as that suggested by Heinke. Two ways of accepting a hypothesis are again suggested--by testing and by proof. As well, the importance of counter-example (Dawson's refutation) is underlined.

The above theories, in combination, isolate at least four processes important in mathematical problem-solving. These are the ideas of conjecture, of verification, of awareness or sensitivity to a difficulty, and of transformation. Table 4B summarizes these ideas. Polya's model is summarized in Table 4A in an attempt to clarify the analogies between the two models. These have already been discussed in the pages just previous.

The Definition

For the purpose of this study, creative thinking will be defined as the sum of four processes--the process of becoming sensitive or aware of the problem situation, the process of

TABLE 4

POLYA'S MODEL AND THE EXPERIMENTAL MODEL FOR CREATIVE THINKING

TYPES OF PROBLEM SOLVING	(A) POLYA'S MODEL-STEPS IN PROBLEM SOLVING	(B) THE SUGGESTED EXPERIMENTAL MODEL FOR CREATIVE THINKING
problems to find: Unknown	I. Understanding the Problem – interpreting the data	SENSITIVITY - An awareness of a problem - Ability to sense implications of data for a possible result, ability to sense deficiencies, gaps in information, missing elements.
Connection Condition Data	II. Devising the Plan A. Varying the Data Decomposing and recombining Generalization Specialization Going back to definition Analogy Recursion-induction Auxiliary elements, problems (includes similar figures, easier problems, diagrams) Working backward-assuming problem is solved. Abstraction	REDEFINITION - rephrasing physical situation in mathematical terms rewriting a problem, re-structuring using an approach not usually suggested.
	B. Using the Data Trial and Error The bright idea	CONJECTURING HYPOTHESES Trial and error Deductive

TABLE 4 (Continued)

POLYA'S MODEL AND THE EXPERIMENTAL MODEL FOR CREATIVE THINKING

TYPES OF PROBLEM SOLVING	(A) POLYA'S MODEL-STEPS IN PROBLEM SOLVING	(B) THE SUGGESTED EXPERIMENTAL MODEL FOR CREATIVE THINKING
	'algebraic' use of a method	VERIFICATION Trial-testing proof counter-example
problems to prove: i. hypothesize ii. verify	III. Carrying Out the Plan	
	IV. Looking back - examining solution to see if proof is most efficient.	
		SENSITIVITY

redefining the given data, or previously known fact; the process of conjecturing possible solutions for a problem; and the process of verifying the proposed solutions.

There is some advantage to using a problem-solving approach to creativity. Torrance defends his use of a definition which stresses problem solving by claiming that his view describes a natural human process which incorporates strong human needs. An incompleteness or disharmony arouses tension, discomfort which the individual attempts to relieve by investigating, manipulating, and making guesses, then testing these guesses, modifying and retesting them. The tension is relieved when the discovery is communicated. He further states:

> There are many other reasons for favoring this definition. It enables us to begin defining operationally the kinds of abilities, mental functioning and personality characteristics that facilitate or inhibit the process. It provides an approach for specifying the kinds of products that result from the process, the kinds of persons who can engage most successfully in the process and the conditions that facilitate the process. The definition also seems to be in harmony with historical usage and equally applicable in scientific, literary, dramatic and interpersonal creativity.
>
> (Torrance, 1967, p. 74)

Freedman's agreement would suggest that this model is valid for a mathematician actually working at mathematics. The criterion of appropriateness is satisfied at this level of the experiment.

The model will be discussed in further detail in Chapter III. The four processes will be defined more specifically, and problems calling for the use of these processes will be constructed.

THE MODEL

I. SENSITIVITY

Understanding the problem, awareness to deficiencies, implications.

II. REDEFINITION

Varying the data-decomposing, recombining, generalization, specialization, analogy, recursion, auxiliary elements, auxiliary problems, working backward, abstraction. Includes the discarding of past experiences when these hinder the solution of a problem.

III. CONJECTURING

Making naive guesses.

Making deductive guesses.

IV. VERIFICATION

Testing.

Proving.

CHAPTER III

THE MODEL

In the review of the literature, four processes have appeared as important in a definition of creativity. These four processes have been chosen to define creativity for the purposes of this study. Briefly, these processes are:

i) Sensitivity: Sensitivity is a divergent process which includes an in-depth understanding of a problem, an awareness to deficiencies, implications, and an ability to extrapolate beyond the obvious.

ii) Redefinition: Redefinition is a convergent process. Varying the data-decomposing, generalizing, specializing, making analogies, abstracting and transforming elements of information are aspects of redefinition. This includes the discarding of past experience when it obstructs the achievement of a solution.

iii) Conjecturing: Conjecturing is a divergent process during which an individual can proceed by making naive guesses or by making deductive guesses. The word "conjecture" has been judged to be more suitable than "hypothesize" so as to keep this process separate from the other uses for "hypotheses and hypothesizing" in the dissertation.

iv) Verification: The convergent process, verification, can proceed through the more naive way of testing or through the more sophisticated method of deductive proof. The method depends upon the individual's state of knowledge and development.

This chapter presents phase one of the study. The aim is

i) to discuss each process so as to establish an intuitive understanding as well as a logical definition for each process.

ii) to present some measures that have been used for such a process, or for a process analogous to it. This is done so that the reader can share the background that resulted in the characteristics established for situations that measure the specific process.

iii) to list some characteristics of each process and to establish some guidelines to the construction of problem situations that require a subject to use the specified process. The constructed problems are then presented.

The model is not an attempt to be comprehensive; the four processes are not thought to explain creative thinking or mathematical problem solving in any exhaustive way, however the four areas specified have been chosen because they describe processes that are important in mathematical activity and because they are of a nature such that they can be understood and implemented in the ordinary classroom by the ordinary teacher.

THE PROCESSES-A DETAILED DEFINITION

Sensitivity

The ability to sense, to be aware of difficulties can be considered as part of understanding the problem. In this sense, it means not only the converging onto the definition and the structuring of a problem but also seeing the possibilities that lie within and beyond the problem.

> Making the familiar strange is a way of shedding preconceptions and perceptual habits. Innocence of vision, a certain naivete, and ingenuousness, characterize the creative individual; if these qualities can be cultivated, the novelty of invention and problem solution should be increased. Problem solving is dependent on this sort of naivete; one of the worst effects of habit is to blind us even to the fact that a problem exists.
>
> (Barron, 1969, p. 133)

Taylor describes this characteristic as follows:

> Ability to sense problems is another intellectual characteristic that is usually included in creativity. It may also lead to motivational features. The capacity to be puzzled may be a very important characteristic. A keen observer once said that part of Einstein's genius was his inability to understand the obvious.
>
> (Taylor, 1962, p. 179-80)

Guilford has tried to define sensitivity in terms of the following items in his battery of tests: (1967, p. 106)

i) Pertinent Questions: A certain action such as setting up a new hamburger stand is proposed. The subject is asked to state four things that should be considered in the choice of the site for the new business venture.

ii) Contingencies: The subject is to "state the conditions which might require the use of specified objects in a described situation." For example, what use might arise for ointment, for pins for two girls berry-picking.

Guilford has some difficulty locating these tests into his model for the structure of the intellect. Originally he assigned sensitivity to the evaluation category; it is now assigned to the category of cognition of implication. This placement defines two characteristics for the process. First, the subject has to go beyond the information given in terms of awareness, effect, or condition. This is the characteristic of implication. Second, the problem must stay within the bounds of things that the subject has experienced in connection before. If the subject has to invent connections, then the process requires more than cognition; it requires some production and some transfer. (p. 106)

Torrance has adapted some of Guilford's ideas for sensitivity or awareness to problems to the level of elementary school subjects in the Ask and Guess Test. This test was found to correlate with other Torrance measures of creativity. (Yamamoto, 1964, p. 76-77; Torrance, 1965, p. 267-295) The task is as follows:

> i) Ask and Guess Test In all forms, subjects are shown a picture (Mother Goose prints for children and certain professional groups, pictures similar to those used in the Thematic Apperception Test for nurses, a picture of boys starting a small business for salesmen and so forth) and given the following set of instructions.
>
> The next three tasks will give you a chance to see how good you are at asking questions to find out things that you do not know and making guesses about possible causes and consequences of events. Look at the picture. What is happening? What can you tell for sure? What do you need to know to understand what is happening, what caused it to happen and what will be the result?
>
> (Torrance, 1966, p. 6)

Torrance maintains that the task examines a subject's ability to sense what he cannot find from looking at the picture and to ask questions that will enable him to fill the gaps in his knowledge.

A study by Rossman indicates that the difference between an inventor and non-inventor is that the latter can only draw out the defects in his environment, the former can suggest what to do about the defects. (Rossman, 1931) Torrance observed these two types of responses during the administrations of his Toy Defects Test. "Many say that the toy dog used in the test should be able to move without suggesting how it would be made to move. Others would put wheels on it, tie a string to pull it along, put a motor or battery on it, place a magnet on its nose, install a winding apparatus and the like." (1965, p. 307)

Adults also show this trait. A group of one hundred eight students were separated into two groups. One group was characterised as critical in motivation, the other group as creative in motivation. All students were asked to read one research article critically, identifying defects in the statement of the problem, hypotheses, data collection, analysis of data and the like, and then to read another, creatively, pointing out the possibilities in these ideas. Each of the reports were rated and then the scores compared to see which group performed better on which assignment. "The students dominated by a creative attitude tend to do a better job in reading research creatively or constructively than they do in reading research critically. The reverse is true of students dominated by a critical motivation. (1965, p. 292) Torrance has also found some evidence (p. 307) that imaginative criticism towards constructive change can be fostered as an attitude. One half of a class of graduate students were asked to read a set of five articles and to critically point out the defects underlying the assumptions, the hypotheses, the statement of the problem and other aspects of the article; the second half were asked to read the articles and point out other possibilities which might arise from the problem, statement of data, and hypothesis. He found that the students in the latter group were much more capable at producing new and novel ideas than those in the former group.

The above data suggests that i) the two predispositions exist in children and in adults; ii) one predisposition or the other may be encouraged by the instruction presented.

Two possible instruction forms to a sensitivity question are suggested by Smith.

i) Formulate as many problems or questions which are suggested by the above data yet are not directly answered by the given information. Include any errors, or shortcomings about which you may feel uncomfortable.

ii) If you can make any changes in or add any additional information to the given problem, indicate what these would be and why. (Smith, 1967, p. 189) The first instruction emphasizes the critical attitude, the second the constructive.

Dr. Freedman of the University Mathematics Department suggested in a personal discussion, that an appropriate situation by which sensitivity might be measured would be an experimental one. An experiment involving some measurement, for example that of volume or area, could be described, followed by one of three possible instructions.

i) Ask any questions which may help you see the relationships which are present in the data.

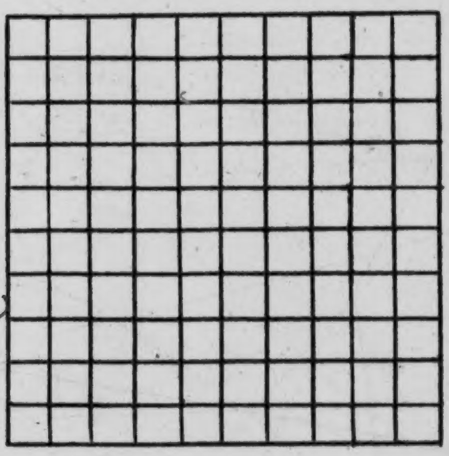
ii) (A possible consequence is specified) What additional data or conditions are necessary to produce this consequence?

iii) What knowledge do you need to show the relationship between consequences and data?

These ideas have been modified in the resulting problem situations. The problem situations following reflect the guidelines and characteristics that are a result of the above discussion. More specific discussion follows the presentation of the problems.

The Problem-Situations.

i) Sensitivity I - The Square



Cut the square in half. What are the shapes of the resulting halves? Draw diagrams to show your reasoning. Feel free to make comments about your thinking.

ii) Sensitivity II - Area-Increase

Two sides of a rectangle are increased by ten percent. How does the area of the new figure compare with that of the original figure? Draw a diagram. Indicate how you would go about solving this problem. Indicate procedures; it is not necessary to complete calculations.

Guidelines for Sensitivity Problem-Situations.

In summary, the process sensitivity will be described as not only a thorough understanding of what the problem is, but also what it can imply. The suggestion is that sensitivity necessarily implies understanding of the problem but that understanding can occur at a level without the extra awareness of possible implications or difficulties.

Situations which are designed to reflect sensitivity should allow the respondant to exhibit puzzlement, to recognize shortcomings. It was felt however that these deficiencies should not be specifically called to attention by the instruction. In order not to "sensitize" the child in this way, the problem-situations required many possible answers or at least a continuum of answers. As a result these situations may describe divergent production, something which Guilford attempted to avoid. The question suggested by Torrance and the ideas suggested by Smith are also divergent production measures, and are scored using the fluency, flexibility measures. A practical consideration which largely determines whether the problem is divergent or convergent is the extent to which direction of response is indicated to the student. The two situations chosen indicate the final philosophy. There are many ways of cutting a square in half, however the student must realize that the possibility exists. There is a continuum of interpretation for "two sides" of a rectangle, however this is not emphasized in the instruction of the problem. The individual was required to see beyond the usual, but he was not told that there was more than the usual in the circumstance.

In both of the chosen problems, the emphasis lies in recognizing the possibilities in the question and in seeing beyond the obvious. Thereby the non-commital type of instruction was thought more appropriate. It is one thing to ask "List all the improvements possible in the toy dog". This gives the examinee very little insight into the specific responses most desirable, however calling specific attention to alternate ways of viewing one half, or to the meaning of "two sides" would eliminate the "awareness" requirement of the problem.

Specific questions of the type suggested by Dr. Freedman did not elicit much response from junior high students in pilot study work. This question may be too sophisticated for this level of children; on the other hand the lack of response may have been mainly due to the poor quality of the specific items. The suggestion of a specific situation followed by questions about mathematical consequences still must be considered a valid possibility for further problem construction.

During the problem construction, a further characteristic of the chosen sensitivity situations became evident. A situation which seemed usual, "straightforward", yet rewarded the individual who became aware of more than the usual, that raised queries beyond the obvious, required some ability to break a "set". There is a set meaning for one-half, a "set" meaning for two sides of a rectangle. The necessity to break these setsconfounds the sensitivity category with the redefinition category. This latter process however is convergent and requires one particular response which is better than the others. The redefinition process will be discussed further below.

The original intention was to keep the processes as separate entities; however, in the interpretation of the ideas into practical problems the separateness did not always seem possible. These problems are then designed to maximize the use of the sensitivity process, but are not equivalent to the process.

Redefinition

The importance of redefinition in the creative process is implied by Poincaré's statement:

> The mathematical facts worthy of being studied are those which by analogy with other facts are capable of leading us to a knowledge of mathematical law just as experimental facts lead us to a knowledge of physical law. They are those which reveal to us unsuspected kinship between other facts, long known, but wrongly believed to be strangers to one another.
>
> (Poincaré, 1952, p. 35)

Guilford reserves a special category in his structure of the intellect for an ability called convergent semantic transformations or more commonly redefinition.

> One of the creative thinking abilities mentioned, redefinition, is classified with convergent thinking factors, a classification that may seem to be somewhat contradictory, but it is in the row for which the kind of product is that of transformations. Much creative effort is in the form of the transformation of something known into something else not previously known.
>
> (Guilford, 1962, p. 162)

At least two characteristics of redefinition are implicit in the above quotes. The first is the concept of a transformation. The other is the association of elements previously not related. The act of reexamining knowledge, previously gained, reorganizing it, associating elements of this segment of knowledge with elements not usually associated with it leads to the discovery of new ideas. "Intelligent behaviour in problem-solving involves the assembly of habit segments never previously associated." (Youtz, 1962, p. 196)

Youtz focusses on this ability and on the factors which inhibit the redefinition process.

> At present our interest is in why it is so rare that we can get the behaviour segments into novel combinations. The difficulty seems to lie in the fact that these novel combinations are composed of 'habit

> segments never previously associated with each other.' Habits stay in their usual form and resist recombination.
>
> (Youtz, 1962, p. 196)

Youtz suggests that there are two ways in which past experience can interfere with the production of novel ideas. The first he calls mechanization, the second functional fixity.

An individual exhibits the factor mechanization when he persists in applying the same tentative solution to a problem even after continued failure. This seems to suggest that past success established such strong habit that the subject was unable to extinguish or break the habit easily. Functional fixity means that the subjects found it difficult to perceive a familiar object in a new light; they were unable to solve problems which demanded a new use of a familiar object.

Luchins has done considerable work in studying the effect of set and attempting to explain some of the reasons for a development of the Einstellung effect. In his three-volume historical account (Luchins, 1970) of Wertheimer's seminars, he suggests several definitions for the Einstellung effect in an attempt to find the most useful and enlightening way of describing the tendency to establish a set. Luchins describes experiments from which he summarizes factors which maximize and minimize the tendency to establish a set. An experiment such as the one first reported in 1942 exemplifies the concerns and the explorations of Luchins. Samples of both adult and school students (2709 in all) were asked to work out answers for the "water-jar" problems. The problem is as follows: The subject is given three jars as measures. Using any combination of these three jars, the subject is to measure out a required amount of water. An example is given--A = 29 units, B = 3 units, obtain a measure of 20 units. The solution would be to fill the A jar, then to empty from A three quantities of B, i.e. 20 = A - 3B. The series of questions are then presented. (Table 5)

TABLE 5

THE QUESTIONS FOR LUCHINS' WATER-JUG EXPERIMENT

No.	A	B	C	Obtain this measure of water
1	29	3		20
2	21	127	3	100
3	14	163	25	99
4	18	43	10	5
5	9	42	6	21
6	20	59	4	31
7	23	49	3	20
8	15	39	3	18
9	28	76	3	25
10	18	48	4	22
11	14	36	8	6

(Luchins, 1942)

Problems 2-6 can be solved in the same way, B - A - 2C. These problems are to produce a set, or the mechanization. Problems 7 and 8 can be solved using B - A - 2C but can also be solved by a simpler solution A - C; problem 9 can be solved by only the shorter method; problems 10 and 11 can be solved either the short or the long way. The control group was given problems 1, 2, 7, 8, 9, 10, 11. The experimental group was divided into two groups, one-half of which was cautioned carefully "DON'T BE BLIND"; the other half was not given any admonition. The first two problems were done and the method verbalized (B - A - 2C), then the rest of the problems were done in order. The results showed that the control group was able to solve item 9 and that all but a few solved 7, 8, 10, and 11 using the shorter method. In contrast, 50-90% of the subjects in the "plain" experimental group failed to solve item 9; 75% of this group used the longer method to solve the items 7, 8, 10, 11. The "DON'T BE BLIND" admonition lessened the percentage of individuals failing to solve item 9, and decreased to 50% of the total number of students using the longer method to solve the other questions. Little relationship was found between establishment of set to age, educational level, or IQ.

Maximum Einstellung was obtained i) by presenting the problems as a stressful speed test; ii) by increasing the number of set-creating problems to a certain limit; iii) by telling or helping the subjects to generalize the rule or algorithm which solved the problem; iv) by using very complex problems which emphasized the use of the algorithm. Minimal Einstellung effect was achieved by the following procedures: i) discussing the possibility of developing the set, and by having similar problems presented to individuals participating in one previous set-establishing situations; ii) adding superfluous information so that required data had to be chosen before the problem was to be solved. (This forced an inspection of data rather than just an application of an algorithm.) iii) preceeding the experiment with exercises in which subjects created problems of the type used in the set-establishing situations; iv) using mixed problems and giving instructions to treat such problems individually, by not generalizing to a rule; v) using other unrelated types of problems between problems, for example, the inclusion of maze and anagram problems among the water-jug problems; vi) requiring all the subjects to try a series of possibilities for each question. Most of the factors which maximized Einstellung were influenced by habits formed during the formal classroom hours and previous learning experiences. (Luchins, 1970, Vol. III, p. 25-30)

Although these studies show that past experience can hinder the transformational process, and thereby reduce the possibility of a novel solution, it would be unwarranted to conclude that the creative individual is the naive individual. The studies on fixation seem to indicate that this rigidity is counteracted if experiences are presented in a variety of ways.

Point vi) is echoed by Polya when he suggests a number of techniques that can be used in approaching a problem.

It is probable that in mathematics a student who is made aware that a number of attacks upon a problem are available, and that even a single problem may have a variety of solutions, can counteract the effect of "set" and remain flexible in his approach.

These suggestions have implication for the redefinition situations to be constructed. It will be necessary to establish a series of situations that determine a set, then to present a situation which can not be solved by the set condition. Maximal set will be established by presenting a number of situations which can be solved by one algorithm, then presenting a condition for which the algorithm cannot work. Since time will limit the number of questions which may be presented, set may be maximized by choosing problems for which students have learned algorithms. The fourth condition listed by Luchins, that of using complex problems, will not be used. The idea in this study, is to see how resourceful Grade nine students can be with the basic geometry concepts they learn at the Junior High School.

Guilford uses the following tasks to measure his factor of redefinition.

i) Gestalt Transformations. The subject is asked to select one of five objects that could be used in whole or in part to accomplish an unusual purpose. For example, "Which object can best be adapted for starting a fire? a) fountain pen b) onion c) pocket watch d) light bulb e) bowling ball.

ii) Object Synthesis. The subject is given two common objects with which, by combination, he is to make something useful. For example, a shoestring and pliers can be used to make a pendulum.

iii) Picture Gestalt. A photograph of a room filled with the objects commonly found in a home is presented to the subject who is asked to suggest what object he would use for accomplishing each of several purposes. For example, "What would you use to tie your hat in the rain?" (Guilford, 1967, p. 181)

In the above situations most people have preconceived ideas on starting a fire or on uses for the various objects Guilford suggests. The set that has to be broken has been established by past experience. In the mathematics problem, mathematical experience not being as common as experience with pliers or onions, the desired set may have to be established more directly.

The process of redefinition may be closely related to the concept of reversibility. Reversibility, as hypothesized by Piaget, refers to the ability of negating classes (inversion) or the ability of negating the relations between classes (reciprocity) in order to build new relationships. (Piaget, 1957, p. 28) O'Bryan (1967) comparing the performances of eighty-five eight year old boys concluded that reversibility was the common denominator exhibited by the creativity (Torrance), the Intelligence and the Piaget-Inhelder tasks. The Piaget-Inhelder Tasks in Flexibility in Hindsight and Foresight require the student

i) to provide a reclassification of an existing arrangement on the basis of the addition of a new element.

ii) to provide as many reclassifications as possible, given all elements initially. After each classification is made, the elements are scattered and the subject is required to make another dichotomy.

iii) to provide a verbal statement from the subject of the classification that he is going to make. (O'Bryan, p. 26)

The Torrance tests used were Unusual Uses, Picture Completion, Picture Construction and Circles. Each task requires the student to provide as many as possible different responses to a certain stimulus. These tasks, as do the Piaget-Inhelder tasks, require the student to abandon or modify a certain approach in order to produce a second one of different emphasis. The approach used in the above tasks differs from those used by Luchins in the number of responses that can be provided. Luchins' problems require one answer, the tasks described by O'Bryan ask the student to provide many responses within the context of the situation. The factor of evaluation or choice as to "correctness" must be applied to a lesser degree in the latter. From this point of view, the tasks used by O'Bryan match more closely the concept of sensitivity as it is used in this study.

How does the concept of redefinition differ from the concept of variety or flexibility? (The following ideas should also distinguish between redefinition and sensitivity.) If past experience has emphasized one certain solution in the subject's mind, such that the subject must use a transformation to overcome this bias before he can produce a correct solution, redefinition is required. Variety refers to the ability to produce many different solutions to a problem. Although this requires the individual to discard previous categorizations or approaches in solution, a particular bias need not have been established. The degree of energy required to overcome a specific orientation is thus reduced, and this energy may be used to produce varying responses. For example, in the water jug problem, if items 2-6 (see page 65) are not presented, finding a variety of possible solutions to item 8 would be a matter of flexibility. If problems 2-6 are included and the specific solution emphasized, then a redefinitive process is required in order to produce other solutions beside the emphasized one.

Redefinition, in this study, will be defined as a convergent process. Questions will follow the guidelines established by Guilford and Luchins as ones for which interpretations or solutions toward which there exist a certain bias have to be discarded in order to produce more appropriate solutions. It is assumed that there is a difference between the mental attitude established on being asked to overcome a predisposition to a problem in order to produce an appropriate solution from that established on being asked to produce as many possible reclassifications given one stimuli. The former will be used in redefinition, the latter will be used in sensitivity.

In summary, redefinition is defined as

i) the act of re-associating and recombining previously unassociated elements of knowledge to result in new knowledge.

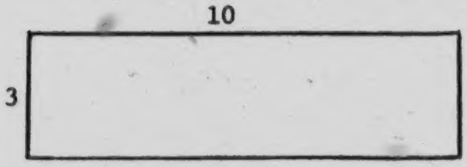
ii) the act of discarding a previously consistent and appropriate technique or approach to similar but different problems in order to facilitate the perception of a new approach to the problem.

The Problem-Situations.

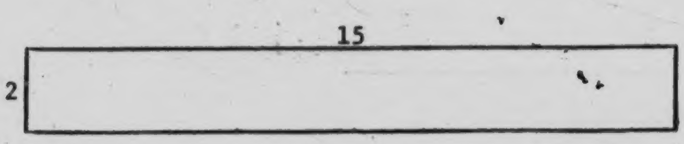
The following problem-situations were chosen as a result of pilot work with situations that reflected the above discussion.

i) Redefinition I - The Board and the Hole

Cut the board into two equal pieces that will cover the hole completely. Show all your attempts including your incorrect ones.



BOARD

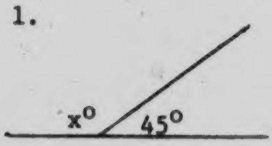


HOLE

ii) Redefinition IIA - Angles

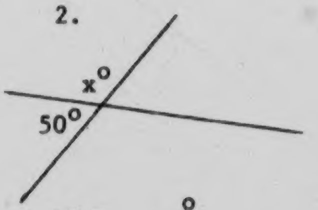
Find the measure of the indicated angles. Show your work. Drawings are not necessarily to scale.

1.



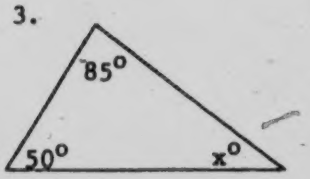
∠x = ___°

2.



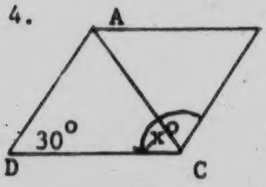
∠x = ___°

3.



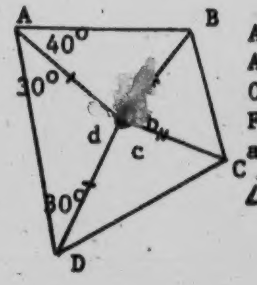
∠x = ___°

4.



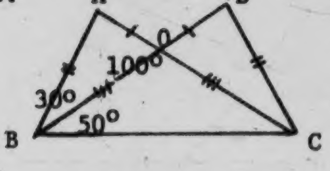
∠x = ___°

5.



AD ∥ BC
AO = OD
OB = OC
Find the measure of the angle formed by
∠(a+b+c+d) = ___°

6.

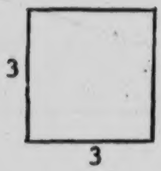


∠DCB = ____°

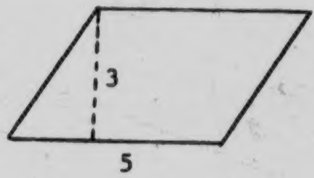
Redefinition IIB - Areas

Find the areas of the following figures. Show your work.

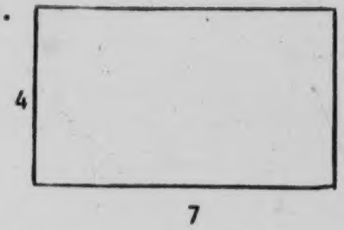
1.



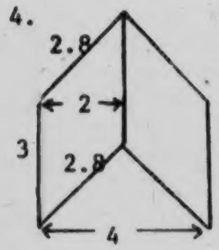
2.



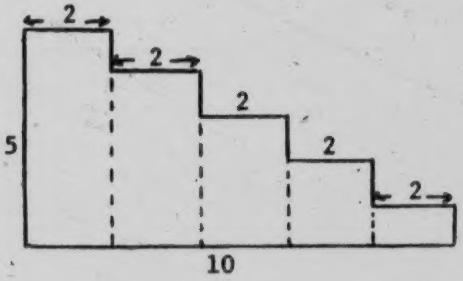
3.



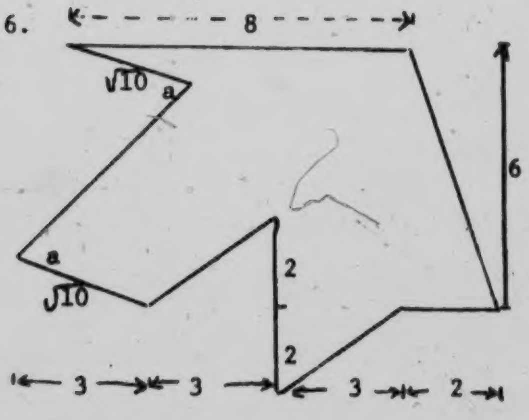
4.



5.



6.



Guidelines for Redefinition Problem-Situations.

The problem-situations which reflect the process of redefinition were constructed according to the following two principles:

i) A pattern for solving a problem, for organizing data or for reconstructing problems is established.

OR

Objects or situations which are commonplace and have specifically defined uses are used in the structure of the problem. The set is then established by one of these two means.

Redefinition I - The Board and the Hole - meets the second half of this requirement. The problem assumes that most individuals have established a definite orientation for the meaning of one-half. It is this orientation that the individual has to break before he can find the specific cut which is specified by the conditions of the problem.

The set in Redefinition II is established in the problem. The first three items in Angles are solved by use of the algorithm (180 - x). These three items are reiterations of the type of question often solved in the classroom with the use of the same algorithm. Similarly, in Areas, the first three items are solved by the formula $A = lw$. Again they reflect the type of question solved in the classroom. In most junior high classrooms, finding area is limited to these regular quadrilaterals, and the use of the generalization, $A = lw$, is emphasized. The pattern in the first three items of the problem-situation is an attempt to reinforce this set.

ii) The subject will be called upon to perform one or more of the following tasks:

a. to combine two or more previously unassociated elements to produce a new combination.

b. to solve a problem which is similar in appearance, format or situation to one just previously experienced. (Refer to water jug problem, page 65)

The subject working The Board and the Hole is asked to cut the board in half but in a way which meets the special conditions of the 15 x 2 hole. There is one answer; the individual is forced to break a set in response to his conception of a certain specified goal. The specified requirement this may establish a set more firmly than an open-ended situation.

Redefinition II was constructed in the mode suggested by Luchins' water jug problem (see page 65). As stated before the first three items reinforce a set towards a certain solution. In Redefinition II, Angles, the established algorithm (180 - x) can be used for item four, is difficult to apply to item six, and does not work for item five. Items four and six are more appropriately solved using alternate procedures. Similarly in Redefinition II, Areas, items four and five can be solved using the algorithm, A = lw, although a more efficient solution can be obtained by first rearranging the pieces, item four into a rectangle of dimensions 3 x 4, item five into half of a rectangle of 5 x 12. Item 6 was very difficult to solve by use of the direct formula to segments of the total piece, however was easily solved by reconstructing into a parallelogram.

Conjecturing

The making of conjectures and the testing of these conjectures are central to most any discussion of problem solving. The individual, involved in the creative act must not only feel the presence of a deficiency or inadequacy, but must be able to respond by offering possible alternatives or suggestions in an attempt to satisfy such an incompleteness.

Smith, who focuses on four aspects of creative problem solving--recognizing and defining a problem, fact finding, hypothesizing, and verifying or elaborating a solution--comments on the third stage in his model.

> The hypothesizing stage of creativity is undoubtedly the most puzzling, the most unexplainable. It involves spurious unpredictable behaviour on the part of the highly creative individual. He may hypothesize a solution to the problem which appears to be irrelevant to the facts. Intuition, feelings and a personal whim may give stronger direction to the creative problem solver than logical deductive reasoning.
>
> (Smith, 1967, p. 121)

The suggestion of an intuitive, almost mystical experience with ideas is echoed by Poincaré in his account of personal creative moments, when he recalled the illuminating crowding of ideas as a driving and independent force of the unconscious. (Poincaré, p. 36)

Dawson (1969) in his analysis of Popper, Polya, and Lakutos concludes that conjecturing is of two types--the naive guessing, and the deductive hypothesizing. Naive guessing occurs when an individual perceives a pattern in his data and extends it in the hope that it will describe the total data. This is Polya's trial and error approach to looking for solutions. The algebraic use of a method as an approach to finding a solution is Dawson's deductive guessing.

Deductive hypothesizing would occur when an individual perceives a pattern or formulates a theorem as a result of some carefully organized experience, possibly some analogy, or some testing. For example, Sawyer (1964) suggests teaching children some of the concepts behind algebraic equations by using some experience which is common to them, such as the trick problem "Think of a Number". Sawyer substitutes a bag for the unknown number, (this bag easily opens to form an X), and proceeds to explain the algorithm behind the trick.

Think of a number	x
Add 3	$x + 3$
Double the result	$2\ (x + 3)$ or $2x + 6$
Take away 4	$2x + 6 - 4$ or $2x + 2$
Divide by 2	$(2x + 2)/2$ or $x + 1$
Take away the original number	$x + 1 - x = 1$

(Sawyer, 1964, p. 64-65)

Sawyer uses the step by step analysis to obtain the algebraic rule for the problem, he leads the pupils through a well-defined procedure which gives them a rationale behind the rules of operations for the numbers. (The naive guesser might instead suggest that the trick lies in the two steps, think of a number and take it away;

double the number and then divide by 2 rather than follow the deductive argument.)

What is a conjecture? The discussion above leads to the following statements which may describe the meaning of this term.

i) A conjecture is a guess on which action can be based.

ii) A conjecture is a statement which seeks to explain a relationship, a statement which attempts to relate aspects of data or knowledge.

iii) A conjecture is a statement which explains the consequences of a certain change, draws a conclusion by relating, generalizing, explains by drawing an implication from one to another.

iv) A conjecture is a statement defining a problem, by putting forth some explanation or description of phenomena later to be verified or discarded.

These four statements describe the type of statements required as responses from students. Situations eliciting this type of response have been constructed by Torrance (1966), Evans (1964), Heinke (1953) and Taylor-Pearce (1971). Some of these situations have been reported below since they have helped to establish guidelines for the problems constructed for the present study.

The Torrance tests have concentrated on the ability to hypothesize consequences and causes.

i) Ask and Guess Test - The situation described for sensitivity was given and the students were asked to provide (a) as "many possible causes as you can of the action shown in the picture."

(b) as "many possibilities as you can of what might happen as a result of what is taking place in the picture." The extra warning "Don't be afraid to guess". is included for both questions.

Evans (1964) constructed sixteen tests to measure divergent thinking and administered these to a total of 91 grade five, six, and seven students in the Michigan school system. The sixteen tests were marked for fluency, flexibility, and originality. Most of the tests were found to correlate with each other as well as with achievement, intelligence, and creativity as measured by the Torrance tests. The following three tests were found to be most reliable (the three levels were compared, and then compared to the test scores available from a previous administration):

> i) I. Finding Similarities (Numbers). In this set of exercises you will be asked to examine sets of three numbers to find as many similarities between the three numbers as you can. That is, you are to discover common properties that these numbers possess. List these common properties briefly but clearly in the spaces provided under each set of numbers.
>
> (p. 98)

> ii) M. Discovering Differences in Sets of Three Numbers. In this set of exercises you will be asked to examine sets of three numbers to find out in what way or ways each of the numbers differs from the other two. Remember that any one of the numbers may differ from the other two in more than one way. Be sure to look for all possible differences and list them in the space beneath each set of numbers. Give your statements briefly but clearly.
>
> (p. 109)

> iii) N. Completion Sentences. In this set of exercises you will be asked to complete sentences which have been begun with some mathematically-related idea. Use the ideas suggested in the beginning of the sentences and experiment to get ideas of how the sentences can be completed correctly. There is not just one way of completing each sentence. In the space beneath each

> sentence, list as many ways of completing the sentence as you can find--even those which are so obvious that you might hesitate to write them.
>
> (p. 111)

Evans' questions although mathematical are similar to that of Torrance. Evans presents a numerical or geometrical situation, then asks the student to provide as many as possible results of a certain nature.

The second test in the study had the same situation presented as I. Finding Similarities but used the instruction "write as many different" responses as possible. Evans found that with students of the junior high level, there was no significant difference in flexibility scores between the two tests, but contrary to expectation, the flexibility scores on the second test were actually lowered. Evans states "The suggestion here is that when students are "forced" to be flexible, as in Test B, they may in actuality be less flexible than when they are allowed to concentrate on fluency of ideas. In effect, they may concentrate so hard on getting different kinds of ideas that they fail to produce many of the relatively simple responses." (p. 129)

The effect of instruction on creativity scores has been raised by individuals such as Torrance (1965), Feldhusen, Treffinger, Van Mondfraus, and Ferris (1971); Manske and Davis (1968), Adams (1968), Christensen, Guilford & Wilson (1957), and Kogan and Morgan (1969). The Christensen study concluded that when individuals were asked to produce "as many clever responses as possible" they produced more clever responses but fewer total responses than those just asked to produce as many responses as possible. Manske and Davis varied the responses four ways, one by asking for original responses; second by asking for practical responses; third for original and practical responses; and fourth, by asking for wild responses. In each case, the responses met the requirement. The greatest number of total responses were made in answer to the fourth instruction. Feldhusen, et. al. (1971) and Adams (1968) found that the number and quality of creative responses increased when the situation was made less competitive, less stressful. The Torrance creativity scores contributed greater predicative power to mathematics achievement of grade nine students if the students were allowed to take the creativity tests home and work on them over a period of four days as compared to standard administration, gamelike administration situations, or alloted incubation periods during administration.

The problem situations, constructed to elicit hypothesizing must be constructed so as to minimize the stressful situation, yet so as to maximize the quality of response.

Taylor-Pearce (1971) constructed problems to measure divergent production of grade eleven students in mathematics, and classified these problems into the categories established by Guilford in the Structure of the Intellect Model. Questions measuring production of units, production of implications were worded "write as many responses as possible"; those measuring transformations and classes required "as many different responses as possible". Following are several examples from Taylor-Pearce's collection of items. Notice again the presentation of data, an indication of the result desired from a combination of this data, then the instruction which calls for many responses. The problem is open-ended; the indication of the result only indicates a direction to the student.

> i) Units. Write as many mathematically true statements as you can about a Shola in the sense defined below:
> A Shola is an odd integer divisible by 39.
>
> ii) Classes. Invent as many systems of equation as you can such that the solution set of each includes the number (1, 2, 3). Try to make the systems as different from each other as possible. When you have thought out a pattern for making sequences, give two or three examples of the pattern, and group the similar systems together. Then look for a different pattern and group in a similar way. Please indicate the groups of systems that are different.
>
> iii) Systems. Show that $1/\sqrt{2} = \sqrt{2}/2$, using as many distinct modes as you can.
>
> iv) Transformations. Imagine that you wish to explain to Grade Six students why we cannot divide by zero. Think out some unusual mathematical approaches that you can use. List as many of these approaches as you can.
>
> v) Implications. Suppose that you are working in a system in which it is true that 2 x 8 = 4. Think out mathematical statements that would be true in this system, and in each case explain briefly (as far as you can) why.
>
> (p. 47-49)

At this point it was decided that maximum response would result from the instruction "Make as many conjectures as you can". The conjecturing questions to be constructed were not to emphasize transformation as this process has been considered under sensitivity and under redefinition. Secondly, as stated by Evans (1964, p. 129; see page 79 for discussion), also by Wallach and Kogan (1967, p. 22) and found in some personal experience, asking junior high students for as many different solutions as possible may cause a hesitant student to apply too great of an evaluative filter. Such a student would not record many hypotheses because he would not consider them sufficiently different from a previous idea. The result may be a discarding of potentially creative ideas by the student himself.

The tendency to repeat many ideas of the same kind would be countered by a class session with the subjects just prior to the administration of the items. In this session, the students would be told that creative individuals produce not just many ideas but ideas which differ from each other and ideas which are unique to the group.

The process of conjecturing requires motivation. The situation must either arise from the individual himself or be sufficiently open to allow many individuals to find interesting aspects which will motivate them to generate ideas about conditions, consequences or implications possible in the given data. Heinke (1953) states that generation of new hypotheses can proceed from old ideas by means of variation--by deletion of data, by addition of data, or by substitution in the data. Whether this variation occurs in the condition or the conclusion of the initial idea, the point in emphasis is that the situation presented to the student must allow for a maximum of this type of variation.

Should the problems include an example of the type of response required? The citing of the example alleviates the possibility that a problem may "leave a student cold", and unsure of the type of response to give. The example clarifies the instructions. This is particularly important for the junior high school student who often lacks confidence, and refuses to answer unless very sure that he is supplying an appropriate response. It was decided to include the example of an appropriate response, even though this example might influence the susceptible student towards the certain mathematical concept area reflected by the exemplar.

Another source of difficulty is in the time limit imposed on the test. Feldhusen, et. al. (1971) found that the larger number of response categories were given later in the administration period. To allow the mystical experience implicit in illumination and the "bright idea" to occur, the evaluation period must extend over a comfortable period of time. It would appear that any test situation must necessarily tap only superficially the subject's conjecturing process. This condition will hopefully be alleviated because the test situations will deal with subject matter to which the children have had exposure. The test situation should trigger any unconscious questioning or supposing that may have been initiated during the previous discussion of the material.

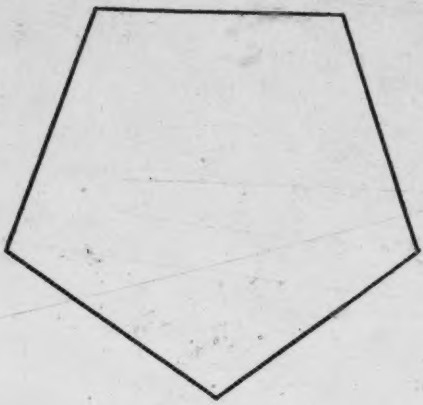
The Problem Situations.

i) Conjecturing I - The Pentagons

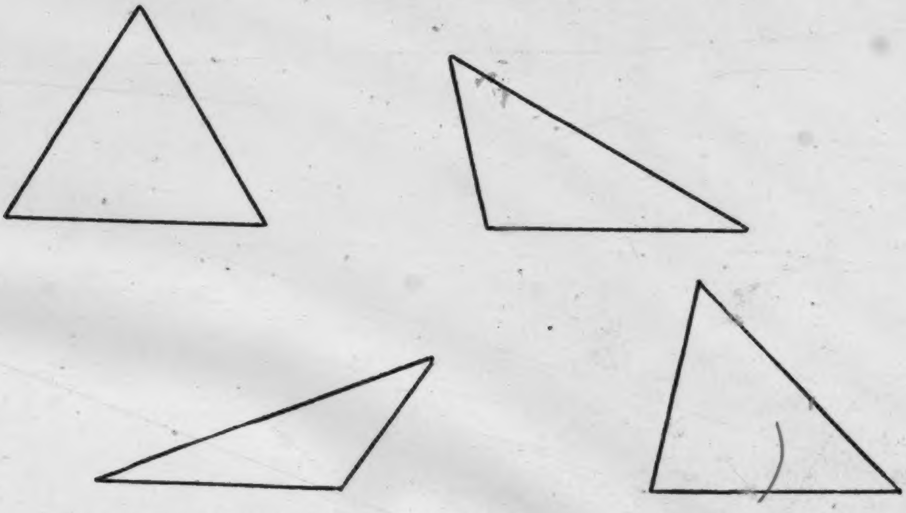
You are given the following shape. Make as many conjectures as you can about the given shape. One example is the following:

> A series of pentagons cannot cover a flat surface without leaving gaps unless the pentagons overlap.

You can use this statement and vary it to make your own conjectures. Then make some of your own.



ii) Conjecturing II - The Triangles



The above is a sequence of triangles. Make some suggestions as to how these triangles are related. For example

> The area of a triangle increases as the perimeter increases.

Use the conjecture to develop others if you wish. Then state some of your own.

Guidelines for Conjecturing Problem-Situations.

The given problem-situations were constructed on the basis of the previous discussion. In summary, they were constructed with the following characteristics in mind:

i) A thought-provoking situation should be presented. The pentagon was considered a potentially rich source of relationships, although visually void of suggestive ideas for someone unsophisticated in mathematical knowledge. For this reason, an example of an appropriate relationship was considered particularly necessary. The inclusion of diagonals in the pentagon or the inclusion of a circumcircle was considered, but these ideas were discarded because it was felt that limiting the data in this way would make the problem too difficult for junior high school students.

The triangle question deals with data more familiar to students of this level. This sequence of triangles is a much less open situation than the pentagon, however it still provides for many responses. The given conjecture again suggests a possible relationship.

ii) A sample hypothesis is included. It was felt this was necessary so as to compensate for the students' minimal experience with situations of this nature.

iii) Instructions should reflect the emphasis desired. In this case, for the reasons discussed above (p. 81), instructions required "as many conjectures as you can".

Other variations in instruction were possible. The following are some variations that were considered: (a) the situation is followed by a statement. The pupil is asked how one part of the statement may be altered in light of the data, yet not changing the truth value of the other part of the statement; (b) a new condition is introduced to the problem. How many relationships does the new condition alter? In what way?; (c) the problem asks for generalizations. If the problem has set forth some specific instances illustrating a similar pattern, a question concerning the pertinent generalizations may be valid.

The unstructured, wide open situation was chosen in favor of the above suggestions since the students in most cases were naive in the area of conjecturing about mathematics, and therefore it seemed that maximal response would result if the students could choose from the broadest field of responses. The first and second possibilities listed above were considered to require a great deal more sophistication in mathematics than possessed by the average grade nine student. The third possibility focusses on a very specific area of hypothesizing. The chosen format seemed to be the best way to reflect the trait of generating statements which may be theorems, i.e. the posing of questions rather than the consideration of many proofs for a statement. The generation of questions was considered particularly suited to many of the concepts discussed at the junior high level. Posing of questions is expected to be more characteristic than making deductive statements ready for proof or refutation of the child at this stage of development.

Verification

Verification, the means by which a possible hypothesis is evaluated, then accepted or rejected, serves a concluding and summarizing purpose to any collection of data. The contention that verification calls upon creative power is justified when we note that many conjectures which remained conjectures for hundreds of years, resisting the attempts of mathematicians to prove or disprove them. Subbaro, in an educational television lecture (1971, ETV) on number theory, underlined a theorem proposed by Fermat that stated that every number of the form $2^{2n} + 1$ would be prime. This conjecture was disproved many years after it had been stated. Another example given by Dr. Subbaro was on the discovery of a 10 x 10 Latin Square, again many years after it had been conjectured that such a square was impossible to construct and even after a computer had spent one hundred hours trying to find a combination to fit such a square.

Guilford's model (Table 3, see page 38) illustrates the continual interplay between evaluation and the generative aspects of problem-solving. Evaluation occurs at all stages: an evaluation of the information and data is made before the problem is defined. Evaluation of the data available is made all throughout the formulation and solution of the problem; evaluation of possible solutions is made throughout the generation of these solutions; and a final and thorough testing, perhaps proving is made when one solution is decided upon. The evaluative filter may then filter out a new idea on which a new problem may be based. Thus although evaluation is summarizing in character it occurs continually, throughout the total problem-solving situation.

Dawson(1969) suggests two distinct types of verification, testing and proving. Both are characteristic of the naive heuristic and the deductive heuristic. An individual may test his hypothesis using examples, attempting to find the one example which will contradict the hypothesis, each corroborating example adding some credability to the conjecture; or he may attempt to generalize these methods and attempt to show the truth or falsity by a logical analysis. Freedman (see page 45), who claims that verification of conclusions is the stage requiring most ingenuity, agrees with Dawson that this process may proceed by i) the search for counter-examples, or ii) the trial and error search for a proof. This trial and error search for a logical discussion may produce counter-examples or corroborating instances for the theorem.

Because of the continual interplay between the verification process and the conjecturing process, a clear line of distinction had to be drawn before construction of the problems. Dawson describes hypothesizing of solutions by naive or deductive guessing, the actual verification by testing or proving. By this definition the followup of the idea, the insertion of a special case, the analysis, organization and communication of his ideas to justify a statement for himself, is verifying.

It was decided not to draw this distinction too closely. The constructed problems would present a problem and ask the student to "prove" it in the best way he knew. The alternative of suggesting a method of and requiring elaboration was not chosen for two reasons:

i) This suggestion would not have resulted in the expression of the ingenuity necessary for evaluation. Because the level of mathematics is not complex, it was felt that the method of testing or proof chosen by an individual would reflect more of his originality than the actual manipulation with the proof.

ii) The method of proof chosen by the individual student would illustrate the level of thinking the individual was inclined to produce. This could vary from a very intuitive explanation to a naive presentation of examples to a complete proof. An indication of the assumptions made and questioned by the pupil would be reflected in the type of verification this pupil used.

In summary, the process of verification will be defined by the following characteristics:

i) To verify means to test by use of specific examples in an attempt to corroborate further a belief in the truth of a statement or in search of a counter-example which will deny the truth of that statement.

ii) To verify means to use an explanation, a justification of the statement using ideas which explain the assumptions and the rationale behind the statement.

iii) To verify means to supply a general proof of the statement, using the methods of deduction and logic.

iv) To verify will include the production of the ideas by which the subjects test the statement, formulated in any of the three ways listed above.

The Problem-Situations.

1) Verifying I - The Polygons

Consider the following statements. Prove each statement as best as you can. Show all your work.

a. If the length of a side of a square is multiplied 5 times, the area is increased 25 times.

b. If the length of a side of a triangle is tripled, the area is multiplied 9 times.

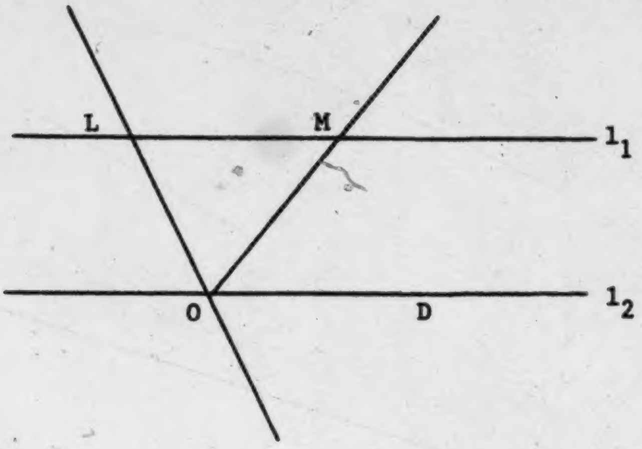
c. If the length of a side of a pentagon is doubled, the area is multiplied 4 times.

d. If the length of a side of a regular geometric figure is multiplied n times, the area is increased n^2 times.

The Problem-Situations.

ii) Verifying II - Parallel Lines

Prove each of the following statements. Show all your work. Diagrams are not necessarily drawn to scale.

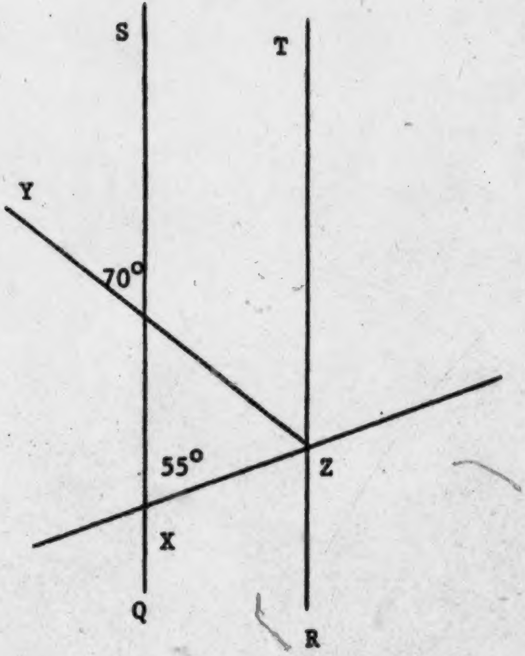


Part A-Abstract Argument

a. $l_1 \parallel l_2$

b. OM is a bisector of ∠LOD if and only if LM = LO.

Prove that this statement is true.



Part B-Specific Argument

If two lines are drawn parallel, QS, RT on the diagram, and the line YZ is drawn to meet QS at an angle of 70 degrees and XZ drawn so that it meets QS at 55 degrees, prove that XZ bisects ∠YZR.

Guidelines for the Verifying Problem-Situations.

The verifying problem-situation presents data, a statement relating the data in conjecture form, and an instruction asking for some form of verification. Alternatively the child may have been asked to supply his own conjecture, and then to examine this conjecture. Asking the student to make his own conjecture for verifying may have established greater motivation, and if the study had been limited to the interview type of format, this alternative may have resulted in greater information. However, because standardization was required for measurement, it was decided to supply the conjecture. In this way, all students started from the same point, and the resulting responses were more easily compared.

Several alternatives were also possible for instructions:

i) Justify the conclusion by as many ways as you can. OR Is the hypothesis true or false? How would you know? Show the truth value in as many ways as possible.

ii) Explain why the conjecture is true or false but do not use specific examples.

iii) The following is one way of proving that the statement is true. Elaborate on this suggestion.

iv) Is the above statement true or false? Prove that it is true (or false) in the best possible way you can think of.

The first suggestion for the task assignment coincides with verification as the hypothesizing of solutions. This emphasizes the open-ended search, a divergent process, and this emphasis was not taken in the study. The emphasis on an evaluative process was desired. This emphasis is given in alternatives two and four; the child has to make a judgement as to whether his justification is the best one he is capable of producing and whether it actually verifies the statement to his satisfaction. Alternative two was not used since it would have eliminated many of the childrens' responses, that of example and counter-example. Dawson's work suggests that this is an important means of verification, especially when the individual is exposed to new information. This would be very important at the junior high school level when youngsters are just beginning to understand the workings of precise mathematics. Alternative four, then, was the desired format used in the problem-situations constructed.

THE PILOT STUDIES

The two problem situations designed to measure each of the four processes discussed in the above chapter were chosen after three separate weeks of experimentation in three Edmonton junior high schools. Problems were written or found and given to students in an attempt to i) determine the type of content that would elicit student interest, ii) determine the type of question that would elicit student response on a meaningful level, iii) gain some insight into the level and type of ideas present in the minds of the students on the subject of geometry.

Pilot Study I

An initial sample of eight problems representing the four processes was given at random to eight grade nine students who volunteered to try the problems. These problems were administered to two students at one time, each student receiving individual instructions and working on a problem that was different from that worked on by his partner. Several objectives were set for this pilot study: i) to experiment with a suitable procedure to use in the study; ii) to gain confidence in an oral type of questioning; iii) to gain insight into the questions that could and that should be asked junior high school students; iv) to determine whether this type of administration of test questions actually supplied information about the thinking of students in addition to that received from a paper and pencil test, thereby justifying the additional time and effort necessary for this type of procedure.

The second objective was strictly personal. Since the experimenter did not have any experience in the oral administration of tests or in the interviewing field, there was a felt need to gain some experience in this type of procedure.

The question of justifying the procedure also seemed important. The experimenter was committed to this type of study, (personal experience in teaching mathematics to students of this age has indicated that strategies are made and acted upon, yet not verbalized to the formal written stage) but did it make sense to use this procedure with a small number of students when the alternative was testing with a statistically significant number of subjects?

There was also the question of timing. The suggested procedure was to let the individual work on a problem for some given time, then to interview him, in an attempt to determine his method of solution. How long should the initial time be? The desired effect was to allow the student sufficient time to get involved in the question, well enough to attempt at least one strategy. The time alloted each question should not be too great. The student should be interviewed while still engrossed in the problem. Retention of problem solving techniques, especially those tried which had been unsuccessful, was going to present a problem at best; the question was how to maximize the amount of information received from the student.

The question of total time also required an answer. How long should the student spend on each question? It was desirable to maximize both the amount of feedback received by each student, and also the number of students which could be interviewed in one day.

The results of this study supported the following ideas:

i) The oral interview was justified. Many responses to problems were not formalized in writing. The method of approach was not clear from written responses even though students were asked to show how they had obtained each response.

ii) The students asked to solve a problem orally without some time to work individually were not able to respond as efficiently as those who had the time to try the problems alone.

iii) The best time allotment seemed to be fifteen minutes of initial pencil and paper individual work, followed by a ten minute interview in which the student explained his approaches to the problem.

iv) The experimental problems should be clear and specific. At this point, it was realized that specific instructions were necessary; students had to know the type of response required. Several of the problems confounded the processes; it was not clear from the responses what means were used to solve the problems. Evidently further experimentation was necessary with the problem-situations. The second pilot study arose from this need.

Pilot Study II

The need to examine a large number of problem-situations in terms of student response, and of clarity of written, oral instruction resulted in the presentation of twenty problems in varying format to three separate classes of junior high students during their regular classroom time. The eight problems presented in the above chapter developed from the twenty used in this administration. These eight chosen to be the measuring instrument elicited the most student response and interest, and best met the definitions for each process.

SUMMARY

In this chapter the model for creative behavior in mathematics was established. The processes of sensitivity, redefinition, conjecturing and verification were defined; the definitions were drawn from literature and from any previous measures established by other authors concerned with creative problem-solving. From these discussions certain conceptions and guidelines for actual problem-situations reflecting these processes were formulated and focussed. In conclusion, two problem-situations believed to emphasize each of the four processes were chosen to be administered to a student sample. Phase two and phase three deal with the administration of these eight problems and with the analysis of the responses to the problem situations.

CHAPTER IV

THE DESIGN FOR PHASE TWO AND PHASE THREE

The study which attempts to provide some insight into the mathematical thinking of junior high students was carried out in three phases. Phase One was concerned with defining a model of creative thinking in mathematics at the junior high school level. This definition included the development of a set of problem-situations which represent the processes describing the model. This part of the study was discussed in Chapter III.

During the second phase of the study the problem-situations were administered to a sample of forty-two grade nine students, the responses to these problems were examined and a scoring scheme developed in terms of the responses. The problems were then scored and the resultant scores analyzed in terms of correlation and image analysis. The development of the scoring scheme is discussed in Chapter V; a detailed discussion of the experimental procedures used for administration of the problems, as well as a discussion of the statistical procedures is discussed in this chapter.

Phase Three of the study is concerned with a non-statistical discussion of the given responses. Oral responses as well as written responses were recorded during the administration. The oral responses provided information important in describing a student's thought processes. Answers to questions such as "What ideas and what cues generate a novel idea?", "What cues or lack of cues result in the absence of ideas?" or "What series of thought passed through the individual's mind as he achieved this solution?" are not clear from written responses. Phase Three involved a descriptive analysis of the responses in an attempt to describe individual thought processes. Chapter VII records some of the series of thoughts so as to provide the reader with some of the approaches and some of the understanding used by different children in responding to a set situation.

ADMINISTRATION OF THE PROBLEM-SITUATIONS

Choice of Sample--Pilot Study Three

Four schools in the Edmonton Public School system volunteered to participate in the study. At this point the experimenter visited one classroom in each school and outlined the proposed study. The students were told that i) the study was to describe creativity in mathematics; ii) the author needed their help in finding out how people solved problems; iii) the process of creativity considered in the study consisted of four things--that of having many ideas in response to a given situation, that of having many ideas, all of different types, in response to a given situation, that of being able to change ideas around and being able to look at things from unusual ways, that of being aware of answers or questions that most persons do not notice, as for example, Pythagorus when he discovered his triangle theorem, Banting when he isolated insulin. Two examples were used to illustrate these ideas. The nine dot problem[2] was presented and the solution discussed with the class. The solution to this problem emphasizes the importance of making many trials in an attempt to find the solution, and that most trials attempting at the solution are similar and that something different, that of going beyond the bounds of the problem, is required before the problem can be solved.

The second example was two sentences which on first glance were meaningless but needed some redefinition of word meaning. For instance,

Time flies you can't they fly too fast.

At this time students were asked to volunteer for the study and the teacher from each classroom was left with the responsibility of providing ten students interested in participating. In two schools, the teachers chose the first ten volunteers, in two other schools the teachers admitted to selecting ten individuals who were interested students, although not necessarily top achievers in mathematics. This was done because it was felt that the more interested students would maintain the work they would miss in their regular classes. (In one of these schools twelve students volunteered and since all of them seemed interested they were accepted.) The main sample then totalled forty-two.

An additional ten students from one school were chosen for a test run before a procedure for administration of items was finalized.

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[2] The problem requires the student to draw four lines but not lifting his pencil, to join all nine dots. . . .

. . .

. . .

During the trial run it was found that hostility towards the experiment developed. There seemed to be several reasons for this hostility and therefore several changes in the administration procedures was deemed necessary.

i) The subjects were of low ability and low achievement. Their knowledge was inadequate and therefore they were unable to handle the problems confidently. The experiment soon became a threat. This was true even though the experimenter tried to supply information needed and to teach the lacking concepts.

This necessity to teach during the time for problem solving resulted in a disruption of the planned time schedule. A standard treatment with this sample soon became impossible.

The schedule was modified to include a session held with the class after a pretest for skills was administered and marked. These tests were reviewed with the students and sources of difficulty were discussed. Problems from inadequate knowledge were thus minimized.

ii) The schedule called for ten minutes of individual work with paper and pencil, then a ten-minute interview during which time the student was asked to recall all his ideas about the question. This probing threatened the individual who felt that his responses had been inadequate. It seemed necessary to encorporate a few additional minutes of further individual work after the interview. The children's need to feel some success in solving the problems could be achieved by giving some directed hints during the last few seconds of the interview and then by allowing the children to follow through with these suggestions. This ensured the students success in finding some solution to the problem. Without this success, their confidence and their interest deteriorated as they proceeded through the experiment.

iii) It was found that the experimenter must be reassuring in his responses to the children's offerings rather than just objective. This again was necessary in order to build up confidence and therefore motivation to continue the next day.

After this experience the final procedure was established.

The Procedures

An achievement of skills test was administered to the students in the sample. The test consisted of thirty short answer questions which covered the basics in geometry required at the junior high level. The questions were strictly recall; the purpose of the test was to establish that the students had a certain level of knowledge. Because of the results of the third pilot group, this level of knowledge was reviewed and emphasized in a discussion of the test during a forty minute class period shortly prior to the beginning of the study. The discussion and test concentrated on the following content--the content also emphasized in the experimental problem-situations.

i) properties of triangles--congruent triangles, number of degrees in the triangle, special properties of the isosceles and equilateral.

ii) parallel lines and the equivalent angles formed.

iii) vertically opposite angles.

iv) area of basic figures.

v) perimeter of basic figures.

The Schedule.

Each student was to work on one problem per day, spending about one-half hour each day on this problem. This would continue for eight days, since there were eight problems. In order to administer the problems in an individual fashion to ten students per day taking into account that the schools could only offer one small room to the experimenter, a staggered type of schedule was designed.

9:00 - student 1   10:00 - student 3
9:10 - student 2   10:10 - student 4   . . .

Students 1, 3, 5, . . . were given the problems in the following sequence: conjecturing I, redefinition I, verifying I, sensitivity I, conjecturing II, redefinition II, verifying II, sensitivity II; students 2, 4, 6, . . . were given the problems in the sequence conjecturing II, redefinition II, verifying II, . . .

Student 1 worked on the problem alone for ten minutes (fifteen minutes for redefinition and verifying), then while student 2 worked individually on his problem, a ten minute oral discussion was conducted with student 1. Then student 2 was interviewed while student 1 was allowed to complete or to add to his initial answers. The oral interviews were taped.

Since the rooms were small, and the oral interview with one student could be overheard by the other student, it was not desireable to have both students working on the same problem. This was the reason for presenting the problems in two different orders. Even with students working on separate problems there may have been some transfer from one student's thought and ideas to the second student's ideas.

On Day One of the experiment, before the problems were presented, an additional five minutes was spent with each student, individually, during which the following ideas were stressed:

i) There may be more than one answer for each problem. Include as many answers as you can.

ii) Guess. If you do not know the answer to a question, guess at an answer that you think is possible.

iii) If you need information, ask the examiner. If this is not sufficient, you may explain how the required information would be used.

iv) Imagine that you are a scientist, or a mathematician. Use the methods that such a person might when trying to study a given question.

The students were asked not to discuss the questions with their classmates until the eight day period was over.

The students were then told that, because it was very important to remember their ideas on how to solve problems, their responses would be taped. An attempt was made to reassure them about the intent of the oral interview--it was only to understand their ideas as much as possible, not "to put them on the spot".

The Oral Interview.

The oral interview was conducted after a period of ten or fifteen minutes of individual work. An alternative procedure would be to ask the student to solve the problem orally at first presentation. This type of procedure is favored by the Russian studies (Soviet Studies, 1969), however was not used in the present study for several reasons.

First, some informal trials with this procedure previous to the study indicated that a long lapse of silence often occurred. The individual type of examination, together with the fact that the examiner was relatively new to the students may have inhibited spontaneity. The student may have felt "put on the spot" especially when the problems did not have immediately obvious solutions.

A period of seemingly random trial and error is necessary on first exposure to a problem. For many people this may be a period of silence. Vygotsky (1962) separated the processes of thought and speech many years ago. Modern technology has shown that the speech center in the brain can be physically separated from the perceptual center, resulting in perception which the individual can reproduce physically, but verbally respond with "I don't know." (Gazziniga, 1967)

Luchins (1970, Vol. III, p. 14) has indicated that set is increased by a stressful situation. This may be even more emphasized when dealing with junior high students who are often hesitant to project and suppose. These individuals often lack the confidence to be wrong.

The above considerations, thereby, resulted in the procedure as outlined. The student was first given ten to fifteen minutes to try the problem by himself. He was then called upon to describe his ideas, and to remember his initial thoughts. He had previously been warned to expect this type of questioning.

The oral interview was considered important for several reasons. First of all students, who operate at the intuitive level with a problem, often do not state their ideas very clearly. It was found during the pilot studies, that although the student response sometimes seemed very shallow, the student's elaboration and oral meandering suggested some in-depth investigation that had not yet reached fruition. Similarly there were times when a written response could be interpreted at higher levels than the student himself realized. The oral interview could serve to clarify the ideas actually present in the student's mind. This information was also necessary in the classification of responses for scoring.

Second, the author wanted as much information as possible about the sequence and relation of ideas in the student's mind as he proceeded to solve the problems. This type of procedure could give in-depth understanding about the level and type of concepts students had in geometry. The type of information which resulted is presented in Chapter VII.

Third, the oral interview could provide information on how the students interpreted the problem-situations. Thereby it would help to determine the adequacy of the constructed situations, as well as to provide some information on the adequacy of the model.

itself.

In summary, the oral questions centered around three aspects:

i) the clarifications and explanation of solution thus far achieved by the student.

ii) the analysis of the responses by the student himself in order to realize some of the influences and relationships which resulted in solutions or prohibited the formation of solutions.

iii) the supplying of hints which aided the student in achieving some understanding of the problem and their solutions. In some cases these questions stimulated the students to proceed in more fruitful directions. The results of the third pilot study indicated that if a student was able to provide some responses that he considered satisfactory, he would continue to be motivated to try the subsequent problems the next day.

The responses supplied by the students after the oral interview were not scored for statistical analysis since the influence of the oral interview on each student was not a measureable one. Although there was a very serious attempt to adhere to the questions planned for the interview and to maintain the same technique from one student to the next, there was an over-riding consideration. It seemed to be more important to elicit maximum information from a given student, therefore additional questions, comments, or supportive responses were given as the interviewer felt it necessary. In addition, there was an attempt to follow a student's ideas or track of reasoning for as long as information was being received. The effect of the discussion, therefore, could not be deemed standard for each student, thus it did not seem that the responses after the interview should be scored.

VALIDITY

Validity is the "process of examining the accuracy of a specific prediction or inference made from a test score," (Cronbach, 1971, p. 443), the degree to which the test is capable of achieving a certain aim. The concept is usually discussed from three views, content validity, predictive validity (criterion-related validity) and construct validity. Content validity is important when studying progress in subject curriculum, the effectiveness of certain training procedures; predictive validity is important in selecting students for college, placing individuals in positions for which their value in the future is judged by some performance in the present. (Nunnally, 1967, p. 83-85) Construct validity refers to a general class of procedures which attempt to measure and describe the attribute that a specific test measures. A construct results from a classification of responses and is a "deliberate creation chosen to organize experience into general law-like statements." (Cronbach, p. 462) Construct validity seeks to explain rather than to indicate a numerical relationship. "The emphasis in construct validity should be on the strength of each relation rather than merely on its statistical significance. (Cronbach, p. 465)

The present study is to be explorative—it does not seek to predict, but to describe constructs which describe creative behavior in mathematics at the junior high school level. In order to be predictive, high correlations would have to be established between the constructed problem-situations and already established criterion tests or describable observations. "Criterion-related (predictive) validation compares test scores or predictions made from them with an external variable considered to provide a direct measure of the characteristic or behavior in question." (Cronbach, p. 444) The key words are direct measure. It is not clear that there exists an accepted measure of creativity in mathematics. The Torrance tests measure some abilities in a general situation, but these abilities may or may not indicate creative abilities in mathematics. The Taylor-Pearce tests for inventiveness were just being written during the construction stages of this study, and certainly, even at the present time, without further validation, could not be considered as accepted direct measures of creativity in mathematics.

The Einstellung materials were developed by Luchins to measure the breaking of a predetermined set. A readaptation of various problems developed by Polya, or found in the Hungarian Problem Series could be used to measure problem-solving abilities, but the level of sophistication of the students in contrast to the elegant mathematics involved in the problems, made this possibility impractical at this level. More important, all these measures are experimental and relate to only parts of the constructed problems. From this point of view, any comparisons with existant tests would describe the constructs of the instrument, and thus fall under the category of construct validity rather than predictive validity.

The problem-situations were compared to other available measures and to each other in an attempt to establish not only what they measured, but the extent to which they were independent of other measures and each other. (Cronbach, p. 467) Scores on the problems were correlated with the scores from the scores on standard ability tests, the scores on achievement tests and the scores on Torrance's creativity tests, verbal and non-verbal. The scores on one problem situation were correlated with the scores from the other problem-situations. These correlations will indicate whether the types of abilities necessary for Conjecturing I, The Pentagon, are different from those necessary for Conjecturing II, The Triangles, or whether they are different from those necessary for any of the other six problems. Another question to be answered is whether the constructed problem-situations call upon abilities measured by the ability or achievement tests?

Some attempt to remove the factor of immediate recall was made when the skills pre-test was administered and returned to the class. The content reflected by this achievement test was discussed with the students prior to the study. Anyone with less than 25/30 on the test was asked to take the paper home and rework the test. This applied to five students out of the forty-two in the study.

In summary the significant correlations between scores can indicate: i) that the processes are not independent entities, ii) that the problem-situations do not exclusively measure only one process but that the solution of one problem required the use of more than one process. As indicated by the discussion during the description of the four processes, (Chapter II) the latter would be expected. For example, redefinition may be necessary in the establishment of a new hypothesis. The problem of establishing the four distinct processes still exists. Do the responses to the problem-situations substantiate the model? This question may be answered to some degree by an image analysis.

Image Analysis

A test score can be described in terms of predicted value p and a non-predicted value e, that is $z_j = p_j + e_j$. The image of test j (p_j) which can be determined from a sample of (n) tests, standardized so that the mean of each is zero and the variance, one, by a linear least squares regression equation of $p_j = w_{jk}z_k$. w_{jj} is defined as zero, and w_{jk} is the regression coefficient for predicting test j from test k. w_{jk} is chosen to maximize the predictability of test j from the sample of n - 1 tests. The anti-image of z is the part of z_j not predictable from the scores on the remaining tests.

If the vector Z represents the tests $z_1, z_2, \ldots z_n$, then $Z = P + E$ where $P = AF$, A is the vector of linear weights and F is the set of image factors.

The correlation can then be found from $ZZ' = PP' + K + EE'$, where K is a set of matrices relating the images of z with the anti-images of z. Although the correlation between p_j and e_j is zero, the correlations between p_j and e_k, and between e_j and e_k, $j \neq k$, are not zero, thus the matrix of correlations (R) can be partioned into the image covariance matrix (G_n) minus the anti-image covariance matrix (Q_n) plus twice the diagonal matrix of anti-image variances. (Kaiser, 1963, p. 158; Guttman, 1953, p. 295) Guttman describes each correlation (r_{jk}) as arising "from two sources: (a) the covariance between the common parts of the two variables, and (b) a special pairwise linkage that may remain between the two variables after the remaining n - 2 variables are partialled out." (p. 289)

The predicted values of z_j are linear combinations of the remaining n - 1 tests; therefore the image analysis yields factor scores that are direct linear combinations of the observable data variables. The common factor space is assumed to be defined by the n variables chosen, and groupings among the n variables are examined by the analysis. This was precisely the need in the present study: to examine the correlations among the test scores in order to see if they grouped in any pattern. This pattern could then be analyzed in terms of the theoretical model. The computing procedures are described by Kaiser (p. 165) and more specifically by Hakstian. Bay (1972, pp. 15, 19, 24)

The Hypotheses

After the problem-situations were administered to all forty-two students, they were scored. The scoring procedures are discussed in Chapter V. Correlations between pairs of the scores for the experimental problems and the established measures of ability, achievement and creativity were also calculated. These correlations were used to answer the following questions.

i) Are the scores for the two problems representing each process significantly correlated? For example, do the scores on Conjecturing I--The Pentagon--correlated with the scores on Conjecturing II--The Triangles?

ii) Are the scores for the problems of conjecturing, sensitivity, redefinition, and verifying significantly intercorrelated?

iii) Are the scores for the experimental problems significantly correlated with the scores for tests of ability (SCAT), achievement (Grade IX standardized test put out by the Department of Education), and creativity (The Torrance Tests of Creativity).

Since it is possible and highly likely that although the correlations between pairs of scores assigned to the problems will be positive, and yet that the four processes defined are distinct, but still required in various degrees by the eight problems, it was important to see if the various factors or clusters of scores did emerge. The image analysis procedure provided information on this question as well as the further question: "If four factors do emerge, are these factors representative of the hypothesized processes, and are these processes distinct from one another?"

The small numbers (N = 41) were not considered a deterrent since a descriptive comparison was required, not a predictive one. No generalization from the sample to a population was made from the rotated image analysis.

CHAPTER V

SCORING PROCEDURES

Chapter V is divided into two major sections. Studies that have been concerned with measurement of creativity in mathematics are first presented, and establish some precedent in the scoring of divergent production. The second section consists of four parts, each dealing specifically with one of the processes. There were two general rationales which governed the scoring procedures. For the question calling for divergent production, responses were judged appropriate rather than correct; for those requiring convergent production, responses were judged correct or incorrect. For the latter a limited number of responses were acceptable, for the former an infinite number.

The conjecturing and sensitivity problems were scored for fluency, variety and novelty. These scores were established from an examination and a classification of the responses. The responses to the redefinition and verifying questions were classified into a hierarchy and each level was assigned a rank. Since all the scores depended upon the responses given, points of procedure between scoring for one process and scoring for another process differed. To make these variations known to the reader, the scoring procedures are discussed in four sections, one for each process. The scoring with respect to individual problems is also discussed wherever more individual variations in procedures were found necessary.

Since the process of conjecturing was to be the main focus in Phase Three of the study, the hypotheses begin by examining the relationship between conjecturing and the other problems. Sensitivity, because it is also an open-ended process was discussed second to conjecturing. This breaks the order used in the model, the order which followed a natural development of Polya's model. The reader, however, must realize that these processes all occur during a creative activity and may occur in any order or in overlapping sequences. Sensitivity is necessary throughout the total solution to a problem. Conjecturing may be a means by which verifying can occur. Redefinition is necessary when conjecturing as well as when verifying.

From this point in the dissertation, the processes will be discussed in the order: conjecturing, sensitivity, redefinition and verifying.

SOME BACKGROUND TO THE MEASUREMENT OF CREATIVITY

A measure of creativity may be contradictory. The idea of assigning a number, no matter how wide an interpretation this number may have, to an act of the mind which in its ultimate has no bounds may be unacceptable to those most concerned with creativity. However the attempt to describe creativity as a process and to describe it as quantitatively as possible has only been prompted by the best of reasons. This search is underscored by the assumptions that there exists creative potential to some degree in everyone, and that the educational system has the obligation to encourage the development of this potential in the individual.

Guilford has used factor analytic methods on a very extensive test battery which included match stick problems, word associations, practical situations, social and personal situations. Many test items were similar to those used by the armed forces to test personnel for adaptability and resourcefulness in a field situation. A factor of sensitivity, four factors of fluency, (word, ideational, associational, and expressional), two factors of flexibility, (spontaneous and adaptive) factors for redefinition, penetration, and elaboration have been established as important in describing creativity.

Torrance adapted some of Guilford's ideas to school children. His emphasis was on open-ended situations which called upon the individual to supply as many responses as possible to a given situation, and then, to score these situations for fluency, flexibility, and originality. Fluency was the number of responses given; flexibility, the number of classes of responses given; and originality, a measure of how infrequently a response is given.[3]

A limited amount of work has been done in the field of measurement of mathematical creativity. Some of the problems constructed by Evans have already been illustrated in Chapter III (page 78). These problems set a numerical or geometrical situation, then ask for as many solutions as possible. The following is a further example of the type of problem constructed by Evans: (1964,

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[3] The Torrance Tests and Manual for Scoring are available from Princeton Press, Princeton, New Jersey.

p. 115)

> 1, 3, 5, 7, 9, 11, ... is an example of a sequence of numbers.
>
> A sequence (of numbers) is formed according to some rule, and once that rule is known, we can find as many numbers in the sequence as we want to. For example, ...
>
> Now go on and write as many sequences as you can, using different kinds of rules. In each case, think of a rule, apply the rule to 1, and continue to apply it to get the next 4 terms of the sequence. Make a brief statement of your rule in the space provided. Any sequence is permissible if you state a rule showing how you constructed it, and you have complete freedom in making up the rules.

Sequence	Brief Statement of Rule Used
1,_____,_____,_____,_____. . .	
1,_____,_____,_____,_____. . .	
1,_____,_____,_____,_____. . .	

The responses to most items (small variations were incorporated depending upon the details required by the specifics of the test items) were scored for fluency, flexibility and originality in the Torrance sense. The fluency score was obtained by assigning one mark for one response, whether the response was correct or not; the flexibility score obtained by assigning one score to each response of a different form. For the question above one mark was given for each different kind of rule used. For example, "add 5," and "add 7" are one type of rule and would score one for flexibility, while "add 5", and "multiply by 5" would score two for flexibility. The score for originality was based on Guilford's statistical reference; a score of 0, 1, 2, 3, or 4 was assigned to each different kind of response, depending upon the percentage of the sample giving this response. (81%-100%--0; 61%-80%--1; 41%-60%--2; 21%-40%--3; 0%-20%--4)

Eastwood (1965) has done some work on the idea of Problem Construction. His type of question is more complex and less specific than the Evans' items and may be argued to be a part of the factor, sensitivity to problems, or related to the area which Torrance measures by his Ask and Guess Test--the ability to formulate hypotheses. Following is the problem used by Eastwood.

Trieste Problem:

> In February, 1960, a United States Navy Submarine called a bathyscaphe, touched down 37,800 feet under the Pacific on the floor of the Mariana Trench. The submarine, Trieste, was towed by a Navy tug nearly two hundred miles from Guam before starting its dive.
>
> It took the Trieste one hour and fifty-eight minutes to get to the depth of 30,000 feet and after that it went down at a rate of one foot per second until 36,000 feet. From then on the speed was checked to half a foot per second. They reached the soft floor of the trench, the deepest in the world, at 1:10 p.m. To their astonishment, the depth, 37,800 feet, was 1,600 feet deeper than the figure expected. The return trip to the surface took three hours and thirty-seven minutes.

The instructions are to construct as many problems as possible, problems that can be answered with reference to the data given.

Eastwood obtained fluency, flexibility and originality scores for this problem but also initiated two other scales, adequacy and penetration. The two scores correlated at 0.90. The former score was found by applying a criteria to the responses; each response was judged on its mathematical adequacy or appropriateness in the situation. The latter was a scale constructed on the basis of expert mathematical opinion on the degree of mathematical insight required in formulating the problem. For example, questions which could be applied directly to the context without any calculation would be scored zero, problems which used the data to predict outcomes of possible application received a score of four. The above data suggests two other considerations in the assessing of responses, the appropriateness of the solution to the problem as well as the depth of insight shown by a response. Since both correlate highly, the former procedure may be used. The appropriateness consideration was incorporated by Taylor-Pearce in his modification of the divergent scores of fluency, flexibility, and originality.

Taylor-Pearce did an extensive mathematical analysis on the scores of fluency, flexibility and originality for mathematical tests of divergent thinking at the grade eleven level. He established a system of concept sets to overcome the difficulty caused by the hierarchial organization of mathematics. This system of establishing the distinct concepts found in one response and comparing it with the concept set representing a second response established an effective way of judging the number of distinct classes of responses made by any individual. If the concept set of one response was a subset of the concept set of a second response, the first was of the same "kind" as the second and thus only one score for flexibility would be awarded. This modification of the flexibility score, Taylor-Pearce called variety.

One difficulty with the originality score, as it was established for the content used by Torrance, was that of "acceptability of response". To quote Taylor-Pearce:

> One aspect of this procedure, however, that does not appear to have been accounted for by previous investigators, is that it is not only the most novel

> responses that may have statistically low frequencies in a sample. Some responses may appear so insignificant to a student, that he does not bother to write them down, but he may write down responses which indicate that he knows these ideas and more. When most students ignore the insignificant, but appropriate responses, these responses tend to have low frequencies. It was considered that the frequency proportion could only determine upper bounds for the responses, and that if there is evidence that a response is implicit in another response, the response should not be assigned a novelty score higher than the novelty score of the response in which it is subsumed.
>
> (Taylor-Pearce, 1971, p. 77)

This adaptation of the originality score, renamed novelty, prevents the penalizing of the more discriminating student by awarding the very uncommon trivial response a high score.

These procedures for establishing variety and novelty were used by the experimenter for the present study. Some further modification for the classification of responses was necessary, but these were determined by the specific questions and are included in the discussion on these questions.

THE SCORING PROCEDURES FOR CONJECTURING, SENSITIVITY, REDEFINITION AND VERIFICATION

Conjecturing

The scoring procedures, based on Taylor-Pearce's modification of Guilford and Torrance, follow the following guidelines:

A fluency mark is given to measure the number of appropriate responses made by a student. Although the decision as to whether a response is appropriate or not was used very sparingly by the original developers (Torrance used it only in the Unusual Modifications to Toys Test), Taylor-Pearce argued that this decision is very important in measuring a specific type of creativity. It was relevant and important to limit responses to the mathematical domain, and to eliminate responses obviously incorrect. In contrast to the broad responses given in answer to Torrance's creativity test items[4] (reasons as to why the individual is at the pool are elicited from the test subjects), mathematical responses can be judged as appropriate or inappropriate within the more limited content. This qualification was very pertinent to the study under discussion in this text. A few responses such as "It reminds me of a barn, . . . a watering trough, a symbol for luck. . .". were given to The Pentagon. These responses were not considered mathematical and thus eliminated from the scoring process.

In general, all responses were given a mark for fluency except those which were non-mathematical, an incorrect statement of description that should be obvious to a grade nine student, or a repetition of a previous statement. This elimination of incorrect and inappropriate responses resulted in a lower fluency score than would have occurred if every response had been accepted. Since one of the purposes of the experiment was to show that most students can respond to the open-ended situation in some meaningful and creative fashion, this more conservative estimate was considered more desireable.

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[4] For eg. A picture of an elf-like creature kneeling by a pool is posed.

The responses to open-ended questions were also given a variety score. This score, again a Taylor-Pearce modification of the Guilford flexibility score, reflects the number of different answers given in response to a question. Theoretically, the more creative student will be able to supply responses that clearly differ from one another. De Bono defines lateral thinking as the search for a problem through many avenues; Guilford speaks of spontaneous flexibility; Taylor-Pearce summarizes "Flexible thinking implies a shift, a jump, an unconnectedness in ideas while rigid thinking implies a sequence, a pathway, a connectedness in ideas." (Taylor-Pearce, p. 61)

The modification again reflects the nature of the specific content. Mathematics is sequential; some ideas, some conjectures are generalizations or extensions of more specific ideas; some ideas are subsets of the larger more inclusive conjectures. In order to recognize and reward the individual who organized a conjecture which may have included several minor ones, yet not to reward the individual who responded with a general statement, then repeated smaller conjectures which were only specific cases of the larger one, a method of categorization was established. Taylor-Pearce (pp. 63-78) established mathematically that every response could be broken down into statements of one concept each, and that these concept statements not only existed, but that each conjecture could be analyzed into a minimal set of concept statements. Once each response was classified in terms of the concept set, the individual was given a score for every concept set which was not included in a previous one.

For example, the following two statements may appear on the same paper in response to the triangle question.

When the base and altitude increase the area increases.

The bases are the same length.

Statement one could be represented by the concept set $Q_1 = \{$area, altitude, linear measure$\}$; statement two by the set $Q_2 = \{$linear measure$\}$. This individual would be awarded a score of 2 for fluency, a score of 1 for variety, since $Q_2 \subset Q_1$.

The oral discussion helped to determine the level and the appropriateness of the response, because the student was able to elaborate on what the response meant to him. This reduced the error attributable to the marker reading implications into a response or to the marker's inability to understand the ideas present behind a response. A response made by two individuals may sound very much the same, yet be representative of two largely differing thought processes and levels. One example is the following response: "Each of the five points where the pentagon touches the circumference of the circle would be the same distance apart". When one student made this as an observation, all he meant was that the pentagon was regular. This was a fairly low level observation. A second student made the same observation, but was really mentally rewording it to mean that if the pentagon was rotated it would describe a circle and was then intrigued with the question of what relationship existed between the side of a pentagon and the circumference of its circumscribed circle.

The oral interview was used to classify the responses and the additional ideas expressed orally were not recognized by the score. The score was limited only to the written ideas because of the possibility that the interaction of ideas between the interviewer and the student might affect the quality of response.

The novelty score is a statistical approximation to some measure of originality in a response. It is based on the assumption that the more frequently given responses are not as original, as unusual in a creative sense, as those which occur less frequently. The Guilford and the Torrance guidelines used the following categories for awarding scores.

Responses made by ______% of the subjects	Originality score
81-100	0
61-80	1
41-60	2
21-40	3
0-20	4

Several difficulties occurred when a strict application of percentage guidelines was attempted. Because of the small number of students in the sample (42) and because of the diverse nature of the responses, it was found that no one response was repeated by more than 55% of the subjects. Some of the responses, although seemingly different on the surface, resulted from the same type of thinking about the variables. It was necessary then to group the statements into categories. Each category was defined by the statements which it contained and these statements would therein be deemed equivalent.

The following procedure was used in establishing the novelty scores.

1) The statements were analyzed into concept sets.

2) The statements with the same concept sets were checked for equivalence. If one statement followed from the same line of thinking (the oral responses helped to establish this), or they resulted in an identical result then the two statements were put into one novelty class.

The separation of the responses for The Triangles seemed more obvious; that for The Pentagon may need some explanation. The following discussion should provide this explanation for the reader as well as provide an understanding of the rationale behind the novelty category.

The responses from this question seemed to fall into fifteen categories--the main four are elaborated.

I. Variation of the Original Statement. As indicated in Chapter III, a suggested conjecture was given:

> A series of pentagons cannot cover a flat surface unless the pentagons overlap.

Twenty-three of the eighty-five responses were variations on this theme. Some were elaborations, some were extensions, but none included any concepts but that of the lattice in their concept set. Some examples are the following:

1) A series of pentagons and other shapes can cover a flat surface.

2) A series of pentagons can cover a rounded surface.

3) A series of pentagons if not regular, can cover a rectangular surface.

II. Some Basic Variations of the Inverse of the Original Statement. The statements of this category all depended upon the concept "the division of the pentagon", and were non-quantitative descriptions of the polygons which resulted when the diagonals of the regular pentagon were drawn. Several subtypes represent this category of equivalent statements. 1) A series of squares would not cover a pentagon. 2) A series of triangles would cover a pentagon. 3) A pentagon is made from a star and five triangles. 4) Ten triangles are formed by joining a vertex with a side opposite. (meaning at right angles) These statements may not be considered equivalent by the adult or the more sophisticated mathematics student. Each type of response may imply a different avenue of research, not necessarily equally fruitful, however the division chosen by the student seemed random and accidental, was descriptive in nature, and in most cases did not seem to lead or to suggest further research. For this reason, assigning equivalent originality scores to these statements seemed justifiable.

If more time were given to a student to discuss and theorize about the pentagon divided by its diagonals, the various descriptions may separate and may lead to the noting of the isosceles 72° triangles, then to a comparison of the lengths of sides and diagonals. Under the circumstances in the present study none of the responses included in category II suggested even a hint of these ideas.

Individuals who were aware of the possibilities inherent in the pentagon resulting from the drawing of the diagonals usually extended or qualified their conjectures. These more elaborate conjectures were not included in the second category, but in a category which included a smaller proportion of responses, and were thus awarded a higher novelty score.

Several examples of responses excluded from category II are the following:

1) The area of any triangles formed by the two sides of the pentagon and any line that joins two vertices of the pentagon will be the same.

2) The small pentagon inside the large pentagon is not in the same position as the original pentagon.

3) Since ten triangles are formed by joining vertex to side in the pentagon, fourteen triangles would be formed in a seven-sided figure.

4) The area of one pentagon will be proportional to another.

III. Properties of Regular Polygons. These were statements of pure description, statements of obvious ideas or ideas which required little or no deduction, statements of memory or statements which were redundancies. Statements in this category were assigned a measure of zero because they were not hypotheses and thus inappropriate. This assignment only applied to the novelty score. Some examples which define category III are the following:

1) If the sides are equal it looks exactly the same, any way you turn it.

2) Each angle will be the same measure in a regular polygon.

3) All sides will be the same in a regular polygon.

4) A pentagon has five sides and five points.

5) All the altitudes from vertex to opposite base are the same.

6) A pentagon may vary in size, but the shape is uniform.

7) The pentagon may be seen as a 3-D figure.

IV. Variations of the Inverse Statement (II) but with an Ordinal Type of Size Specifications. The size specifications included in the responses from this category were ones of comparison between one or more of the polygons that made up the pentagon. For example:

1) The diagonals drawn result in five isosceles triangles, five obtuse triangles, and a pentagon.

2) The figure is made up of five similar isosceles triangles.

3) A pentagon can be divided into an infinite number of stars.

4) Three is the smallest number of triangles in the pentagon.

Response number four illustrates one of the weaknesses in the classification scheme for novelty. Subjective judgement of the scorer determines the inclusion or the exclusion of a certain response. In this case it may be very justifiably argued that the thinking in response four is on a level separate from the other three. There seems to be a hint of division, or realizing a number of possible divisions, and then of a comparison among classes (of divisions) rather than among units (of one division). At the time of the classification, the experimenter made her decision by listening to the taped responses of the subject, which did not suggest this individual's appreciation of this type of higher implication of his own response. This response was made by one student, and would not change the novelty score for the category, although the subject responsible for the response may have received a more conservative grade than he deserved.

The remaining categories were very small numerically. Category V dealt with the altering of the sides or the shape of the pentagon, Category VI with the relationship of the perimeter and the area of the pentagon, Category VII with the comparison between the measures of parameters of the pentagon or its part. For example:

> The area of any two triangles formed by the side of a pentagon and two lines that join 2 vertices of a pentagon are the same.
>
> When you divide the pentagon into triangles by the altitudes are the two triangles the same size?

The other categories VIII-XVI contained one or two responses in each. One of the most interesting was the following:

> The area of an inner pentagon will be proportional to the original one.

Several of the one-response categories were considered as subsets (had concept sets which were subsets) of the first four categories and were then awarded a score equal to that of the superset.

The novelty score awarded to every response within a specific category was calculated by counting the number of responses in the specific category, then by comparing this number to forty-two as a percentage. Guilford's percentage limits listed above were then applied.

One other qualification in the novelty score was made before the final score was obtained. An initial score of zero was assigned to responses that were redundant of some ideas stated in the question, that were statements of fact or were facts that are commonly given in grade nine classes. Several examples follow

> The total number of degrees in a pentagon is 360.
> The area is 30 square units.
> A series of pentagons cannot cover a rectangular surface without leaving gaps. (This is essentially a restatement of the given hypothesis.)

All other responses were given an initial score of one. The final novelty score was then equal to the initial novelty score $\left[\binom{0}{1}\right]$x the statistical novelty score $[0\ 1\ 2\ 3\ 4]$. This score was awarded to each given response that a student made, and the average of the two highest scores became that individual's novelty score.

Sensitivity

Although both The Square and Area Increase were logically assessed as problems measuring the same trait, the responses given seemed to indicate that the children of this level saw the two questions differently. The similarities and the breakdown of these similarities is discussed under two points: one--the relative open-endedness of the problems, and two--the relative importance of the spatial factor in each problem.

i) Most of the students assumed both problems to have a definite number of answers. This similarity breaks down, however, when students realize that both questions really have many answers. The difference is that the infinite number of responses for The Square are not interdependent or sequential. The responses to Area Increase are infinite along a continuum which can be described by setting limits on both ends. The area tends to zero as the altitude approaches zero, thus the area decreases if the number of degrees between the slant side and the horizontal tends to zero. The greatest increase occurs when both of the two adjacent sides are increased by ten percent. Since these relationships are interdependent and sequential, this question, at a more sophisticated level, becomes a convergent type of problem. The children participating in the experiment, however did not see it at this level; most of them did not even see the problem having more than the obvious two responses of increasing length and of increasing width. (See discussion in Chapter VI)

ii) Both problems had a spatial factor, however the spatial factor played a greater part in the The Square than it did for the Area Increase. Since most students answered the latter problem only by increasing the two lengths or the two widths, the spatial factor did not even enter into their conception. On the other hand, the spatial factor for the former was an obvious one in the thinking for The Square.

Because of these factors, barring the fact that the general procedure for scoring the two sensitivity problems was the same, the novelty categories for the second sensitivity problems took on a more sequential character than those for the first sensitivity problem. To illustrate these facts, a more specific enumeration of the categories used in the scoring follows.

Each set of written responses was scored for fluency, variety and novelty. All responses except those that didn't meet the specifications of the problem (didn't cut the square in half, or didn't increase two sides of a rectangle by 10%) were awarded a score of one for fluency; the total number of responses became the fluency score of that individual. Scores ranged from 1 to 8 for The Square and from 0 to 4 for Area Increase.

The initial fifty different responses given by the group to the first sensitivity problem--The Square--were classified into twenty-three different types of responses. Some of the characteristics used as concepts to establish these categories were i) the straight-line cut, (vertical, horizontal, diagonal, off-diagonal) ii) the zig-zag cut, (vertical, horizontal, diagonal, off-diagonal) iii) the L-shaped cut, iv) the curved line, v) the insert using one side or corner of the square, vi) topological cut, vii) the use of more than two pieces to make a half.

One mark was given to each category represented by the responses made by an individual; in other words, the variety score for an individual indicates the number of different types of responses made by that individual. The variety score for The Square ranged from one to eight.

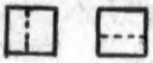
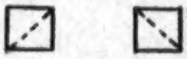
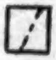
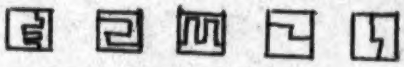
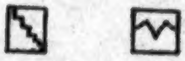
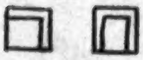
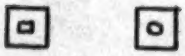
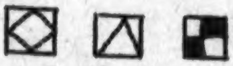
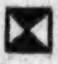
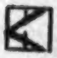
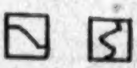
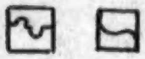
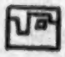
The classification for variety was to be used to establish novelty scores, as had been done for conjecturing. This plan was soon realized as inadequate. Many of the responses fell into category i or ii; these responses would be given a score of zero. Of the remaining responses, those made by one-half of the class were placed in categories iv and vi; these would receive a score of two. All other scores were spread thinly among the remaining categories and therefore would all receive scores of four. A further grouping to reduce the number of categories was deemed necessary if the novelty score was to be more meaningful. The further classification as shown in Table 6 resulted in a greater spread of scores and reduced the number of responses receiving a top score of four to twelve out of the original one hundred fifty-four.

One weakness in the scoring scheme, that of subjective judgement used in the classification of responses for novelty, has already been mentioned in connection with conjecturing. Two other difficulties became evident. First, the scoring procedure does not enable the reader to distinguish between two extremes of response, one--the analytical listing and classification of the possible halves, making use of rough drawing and mental conceptualization of the meaning of the term one-half, as compared to two--the detailed, artistic or very careful drawing of half a square, resulting in a beautiful but single response. Both extremes are statistically novel, yet the first individual wins a greater fluency score, variety score, and sometimes may average a higher novel score.

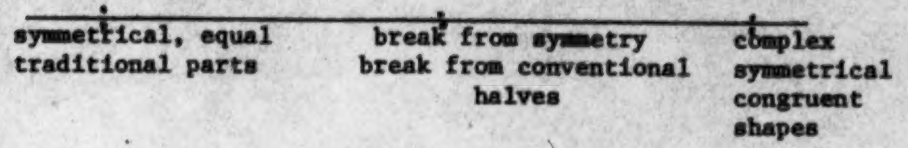
Second, the scoring procedure does not answer the question of whether the symmetrical responses or assymmetrical responses are more creative. Is the break from symmetry a mark of originality or of flexibility? Is it just an avoidance technique which makes the problem easier to answer? Is the imposition of symmetry onto the meaning of one-half a tradition or is it a self-imposed restriction which results in a more aesthetically satisfying response? Possibly responses may fall along a continuum such as this:

TABLE 6

CATEGORIZATION FOR SCORING THE SQUARE-NOVELTY

CATEGORY	CHARACTERISTICS	NUMBER OF RESPONSES
i		39
ii		40
iii	 Considered a subset of 2	6
iv		21
v		17
vi		11
vii	 Variations of shape inside	3
viii	 Variations of inside cut 	8
ix		1
x		5
xi		3
xii		1
xiii	Elaborate symmetrical design	1

BELFAST BOND



Originality is difficult to separate from complexity in this problem. The most complex responses confound spatial sensibility, balance and physical skills. Just how to award scores so as to recognize these problems can be answered only after further research into the types of individuals who give certain types of responses.

As mentioned before the categories for Area Increase are of a type different from those established for The Square because they have a sequential nature. Because of this, the safeguard applied in the other problems to prevent the rewarding of responses that were given infrequently, but that were defined by concept sets which were subsets of concept sets for responses given much more frequently, was not necessary. The categories were as follows:

A) Increasing four sides of the figure maintaining the rectangle.
B) Increasing two lengths only, maintaining the rectangle.
C) Increasing two widths only, maintaining the rectangle.
D) Increasing one length and one width only.
E) Increasing two lengths as one response and two widths as a second response.
F) Increasing two lengths and two widths and one length, one width. (3 responses)
G) Increasing two lengths and one length, one width. (2 responses)
H) Increasing four lengths, one length and one width. (2 responses)
I) Using the square for easy calculation, then generalizing to rectangle.
J) Doing more than one example for each calculation.
K) Attempting to generalize the idea exemplified.
L) Posing completely inappropriate or incorrect ideas.

Notice that categories C, B, and D are independent of each other. $B \subset E \subset F$, $C \subset E \subset F$, $D \subset G \subset F$, and $A \subset H$. Categories I, J, K are used in conjunction with categories A-H inclusive. Thus a response (A, J) which increased four sides of a figure and used several examples to establish a generalization would score two on fluency, and one on variety, since the response (A, J) would subsume the response A, and (A, J) can be considered as a subset of J. (ie. $A \subset (A, J) \subset J$) The range for variety on this problem was zero to three.

The statistics on which the novelty score was based were arrived at by combining the number of responses made in a certain category plus the number of responses made in the categories which superceded it. Thus the percentage from which the novelty score for B is achieved = 1/42 (the number of responses given in B + the number of responses in E + the number of responses in F + the number of responses in G.) x 100. The percentage guidelines stated on page 124 were then applied.

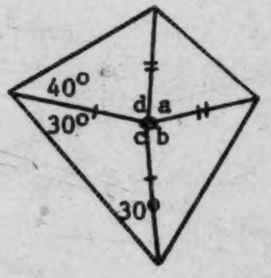
Although categories I-L were methods of verification, it was decided to include them as appropriate responses and to treat them in a manner consistent to that of the other categories. These responses were indications of thorough mathematical investigation, and these individuals felt the need to establish the truth of the certain interpretation they had chosen for the question. As a result, these individuals did not have the time to investigate the other interpretations for the question. To distinguish the individuals who would have proceeded to further investigation from those who would have not seen these other possibilities would require an extended time for the problem. This was not possible here. The awarding of the additional score to students who had included responses of type I-L may be over-rating these individuals for novelty. This error was reduced somewhat by the averaging of two novelty scores assigned to two responses from two different categories to obtain the final novelty score for that individual. The novelty scores for Area Increase ranged from zero to four.

Redefinition

The redefinition problem-situations are to measure the ability of the individual to solve a series of seemingly similar problems by the most appropriate means. The decision as to which response was most appropriate was based on efficiency, that is, which method resulted in the solution in the easiest and quickest way. This decision was based on seeing the problem in isolation, not in the context of any special previous solution.

Each question consists of six parts. The first three parts, which were short answer questions similar to those used in any classroom, were not scored. They, however, served two purposes--one, they established a set, a predisposition towards the problems which the students would have to break; second--they served to build student confidence, or enabled the student to review concepts in terms of a specific situation. The second purpose turned out to be quite important in the experiment since, as mentioned before, the students appeared unable to function well without reassuring feedback.

For the three remaining parts, a list of the possible solutions and the approaches was made. The most straight-forward one was assigned a score of two, the others a score of one. An incorrect or inappropriate answer was scored zero. For example, the answer to Angles, part five, is easily seen if it is realized that ∠(a + b + c + d) is equivalent to a complete revolution.

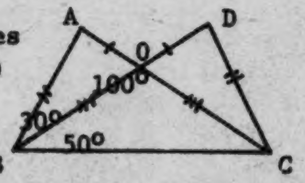


$m(\angle(a + b + c + d)^{\circ}) = \_\_\_\_\_^{\circ}$

Because the previous questions had focused on the algorithm '180 - m (given∠) = m(x), where x is the required angle, many students attempted to find the individual measures for angles a, b, c, and d with the intention of combining them to form the total. The method based on one revolution was awarded a score of two, any other method of solution, if correct, was given a score of one. For this question, no one achieved a correct solution by any means other than the complete revolution method.

The second example illustrates a possible weakness in this question.

Angles
(vi)

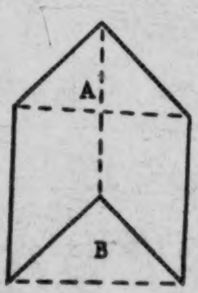


$\angle DCB = \_\_\_\_\_^{\circ}$

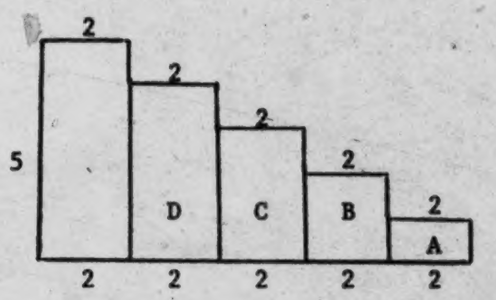
The solution awarded a score of two was dependent on realizing that $\triangle ABC = \triangle DBC$, thus $\angle C = 30^\circ + 50^\circ$. Most students solved the problem by using $\triangle ABO$ and $\triangle ODC$, that is, by analyzing the parts rather than the whole figure.

In this problem, it is not clear that the former solution is better than the second, although it is more direct. This difficulty is indicated further in the analysis of the statistical results.

The second question in the second problem for redefinition, Areas, also consisted of six parts, the first three parts dealing with the areas of three parallelograms, two of them right-angled. These answers provided a setting, emotional and academic, and were not awarded marks. The remaining three attempted to provide situations for which the set would hinder the findings of a solution which was most direct. Specifically part four could easily be solved by moving A to position B, forming a square. This approach was awarded two marks. Two other approaches~~,~~—one of finding the area of two parallelograms, then summing the areas, and second of finding the area of two squares in three dimensional arrangement (an idea discouraged in the instructions but one that tended to reappear), were awarded scores of one.



The favored approach to the fifth part of the problem was again one of rearrangement, that of doubling the given information to form a rectangle of 5 units by 12 units, then realizing the area of the required part was half of 60 square units. A second alternative was to use rectangles A and B to complete rectangles D and C respectively, to obtain a large rectangle of 5 units by 6 units. These two approaches were given scores of two; approaches which utilized the division into smaller parts and their summation, were given scores of one.



Part six of the question required rearrangement into a parallelogram for easy solution. Finding the area by division of the parts was very unwieldy, although one student did manage to get the correct answer in this fashion. The former solution scored two, the latter, one.

The question of whether one solution is justifiably favored over the other still remains. Although rearrangement was a convenient way of solving these specific problems, the approach of division is certainly a mathematically valid and understandable one. Division and analysis by parts is a method of calculus. As suggested by one of the students who examined both types of solution to the staircase—"Doing it by parts was easier. I knew it would work; the other way is faster, if you notice it, but longer since you have to look for it."

This question is of the type constructed by Luchins in his work on the Einstellung effect. The first three parts establish the set for area and the use of formulae. Part five is possible by rearrangement or by division, although less efficiently by the latter; Part six is possible only very painstakingly, by division. As a result, the more fruitful source of information is in the second analysis of this problem--the comparison of individuals who solved this problem--and the greater value of this problem is that of helping students learn to overcome the Einstellung effect.

The total score for each section ranged from zero to six depending upon the solution given by the student.

The first redefinition question was much more difficult to mark since the redefinitive aspect was enclosed in a single complex problem rather than in several parts which required a simple process. Whereas in the second redefinition problem each small part focused on one fixation which had to be broken, to cut the board in half and to fit it onto the hole required the student to break from the traditional straight cut of a half, to manipulate the many possibilities of irregular cuts until the numbers matched the required dimensions, and then to fit these pieces together to form a second complete whole. Thus, the symmetry, the number ratios, and the spatial rearrangement of the pieces were additional demands compounded on the simple requirement of cutting the piece in half, and served as stumbling blocks for at least a portion of the sample.

The original intention was to score the types of responses given by students as they attempted to solve the problem. The children were thus given instructions to include all their ideas and all their attempts even though they realized that they were incorrect. This information was to be important in analyzing the type of thought which resulted in correct answers but it was also thought that it might reveal some numerical relationship between number and type of variations that an individual attempted and the ability to solve problems of this type. The preliminary work soon showed this approach unfeasible. Some individuals only had "to mutter a few incantations under their breath" before they quickly drew the correct response. Others drew many variations of the zigzag cut and even then could not establish the correct connection. Two individuals could approach the problem in similar ways yet one would quickly see a solution, another would not at all. Other factors besides the number and type of drawings seemed to correlate with a solution; these factors would be better analyzed by discussion than by numerical means.

In the last analysis, the best representation of an individual's ability to solve the problem seemed to be the time taken to solve the problem. The subjects had been asked to record the time at which a solution was achieved. A clock was provided and the student noted the time of solution to the nearest minute. Since the time had not been intended as a basis for scoring, extra care was not taken, and may prove to be a source of error in the analysis. The students were classified into categories based on solution within five minute intervals. The following table indicates the scoring.

TABLE 7

THE SCORES ASSIGNED TO THE BOARD AND THE HOLE DETERMINED BY TIME TAKEN TO SOLVE THE PROBLEM

TIME (x) in minutes	SCORE
$x \leq 5$	3
$5 < x \leq 10$	2
$10 < x \leq 15$	1
$15 < x$	0

The assumptions behind this procedure are several:

i) The greater the time taken to solve the problem, the less able an individual is able to break a set to reorganize his ideas, his conceptions, and to adapt them to the given conditions.

ii) The time-redefinition ability proportion lies along a continuous and an interval type of scale. This scoring scheme suggests that the difference between the ability of an individual who solves the problem in five minutes and that of the individual who solves it in ten minutes is the same as the difference between the abilities of the individuals who solve the problem in ten and fifteen minutes respectively.

The difficulty with these assumptions is obvious in terms of reliability. The flash of insight responsible for those responses that took place within the first few minutes will not necessarily happen to the same individual in another context. It is suggested, however, that the individual who scored high in this problem would solve a greater proportion of similar problems if given a series of these problems than would the individual who obtained a low score.

VERIFICATION

The first verifying problem-situation, Polygons, is composed of four parts, the first three comparing the area of a regular polygon to the length of its sides, the fourth stating the same relationship for a general polygon. The polygon varies from the least complex, the square, to the triangle, to the pentagon, then to the general case.

A scoring procedure similar to that used for the second sensitivity problem, Area Increase, was set up. The responses were first awarded a score of one or zero depending upon whether it was correct or incorrect; they were then categorized into sets which were similar in their approach of solution. Six scoring categories were the result and are summarized in Table 8.

These categories indicate levels of answers which can generalize to other situations. The categories that resulted from an organization of the responses to the second verifying problem--Parallel Lines, Part A--are shown in Table 9 and illustrate a similar hierarchy. The responses from Part B separated into similar categories. (Table 10)

TABLE 8

CATEGORIES OF RESPONSES FOR POLYGONS

CATEGORY	SCORE
I) Restatement of Problem-Descriptive restatement or an attempt to justify by restating the problem in a different form.	0
II) a. Use of one example to substantiate a proof. b. Some attempt to generalize by using the side measure of one for the polygon. c. Distinction between side and altitude not realized; the square sometimes seen as a generalized polygon.	1
III) As II) but a distinction made between altitude and side, but no formal treatment made of the assumption that relates side and altitude. b. Use of more than one example.	2
IV) a. Some indication of a general proof. b. An intuitive idea of similar figures used to relate increase of side to respective increase in area. c. Some indication (clarified by the verbal discussion) that a generalization was understood, even though it was not stated in formal form.	3
V) a. A geometric proof was attempted. b. An intuitive geometric idea was generalized. c. Some attempt made to generalize a geometric drawing.	4
VI) a. An unfinished attempt to formulate a hypothesis or theory that would generalize. b. Assumptions behind altitude-side relationship are still not questioned.	4
VII) Use of many examples resulting in a generalized algebraic statement. b. Proof by formula, some discussion of the assumption that altitude is proportional to side.	5
VIII) Proof by formula-discussion and consideration of the assumption.	6

TABLE 9

CATEGORIES OF RESPONSES FOR PARALLEL LINES, ABSTRACT ARGUMENT

CATEGORY	SCORE
I) Restatement of the problem a. descriptive b. an attempt to justify by use of an assumption that simplified the problem, ie. that the triangle was equilateral. c. a restatement in the contrapositive form. There is no realization that these are just equivalent forms of the same statement.	0
II) Proof by measurement or by construction.	1
III) A generalized proof but based on the assumption that the triangle was a specific one--the equilateral triangle.	2
IV) Proof based on construction of the parallelogram and the resulting congruent triangles. No proof of the congruency, the presence of two congruent triangles in a parallelogram justified in terms of observation or intuition.	3
V) Proof based on the construction of a parallelogram with all sides congruent. Justification of congruency supplied in oral discussion when asked on the basis of equal angles, which makes the proof equivalent to the one below; or justification on the basis of three congruent sides of the parallelogram.	4
VI) Proof established from first principles. Equal angles justified on the basis of parallel lines; no unjustified assumptions.	5

TABLE 10

RESPONSES FOR PARALLEL LINES, SPECIFIC ARGUMENT

CATEGORY	SCORE
I) Restatement of the problem a. no attempt at justification b. justification by measuring with protractor.	0
II) Proof by construction--a statement of congruent triangles but no attempt made to describe the conditions necessary for congruent triangles.	1
III) Proof approximating desired proof, but making use of the conclusion to justify conclusion. (But half of 110 is 55, so angle is bisected.) Note the realization of the important facts but inability to separate the given information from the required information.	2
IV) Proof attempted by use of the parallelogram. Argument for congruent triangles based on the construction of arcs with equal radius. A generalization of the problem is made.	3
V) Proof as desired achieved during the oral discussion. Some confusion during the first ten minutes due to inability to read the diagram or to understanding the meaning and implication of the question--eg. the meaning of bisection.	3
VI) Proof based on the result to Part A. Since Part A was true, then Part B, a special case, was also necessarily true. Part A was not justified adequately.	4
VII) Proof by use of vertically opposite angles and parallel lines.	5
VIII) Proof by use of parallelogram, the two congruent triangles and thus the two congruent angles justified.	5

Note that the initial category which was assigned a score of zero was one of reiteration, of restatement, of description and elaboration of the given problem. A second level of proof seemed to concentrate on the use of a specific or single example to illustrate the proof. The use of specific and familiar illustrations of the concept established the idea and was sufficient to justify its truth for the student. A third level was a partial generalization, a justification which went beyond a specific case but in which underlying assumptions were accepted and sometimes not even separated. A fourth level was an unfinished theory or statement which was stated in formal and general terms, which established the necessary conditions and set out the concepts important for the proof, but did not formally justify the assumptions. The fifth level was the formal proof, one in which assumptions were separated and explained.

RELIABILITY

In order to establish the extent to which the scores awarded were capable of being reproduced, a sample of five folders was marked by a graduate student in mathematics education. This person had not been involved in any of the philosophy behind the scoring of responses to divergent thinking items, and was not conversant with the Torrance tests. This second scorer marked the papers according to a close description of the scoring procedures given to her by the experimenter. Table 11 presents the scores given by the

TABLE 11

SCORES ASSIGNED BY TWO INDEPENDENT MARKERS FOR CONJECTURING

Pupil	CONJECTURING I				CONJECTURING II			
	Fluency		Variety		Fluency		Variety	
	A	B	A	B	A	B	A	B
I	3	4	2	3	6	6	5	5
II	3	3	3	3	2	2	1	1
III	2	2	1	1	3	3	2	2
IV	3	4	2	2	4	8	4	6
V	4	4	3	3	6	8	3	4
TOTALS	15	17	11	12	21	27	15	18
% AGREEMENT*	87%		91%		71%		80%	

*% AGREEMENT = (Difference + Total A) x 100

experimenter and the scores given by the second marker for the two conjecturing problems.

The scores given by this individual differed for three reasons: i) An incomplete understanding of the scoring procedures used provided some difficulty in assigning concept sets. This was corrected after further discussion. ii) An incomplete understanding of the students' written responses provides a source of difficulty for any scorer. The oral comments made by the student clarified or biased the experimenter in her scoring of the responses. These oral comments were not heard by the second scorer. iii) There was some disagreement on the appropriateness of responses. The experimenter's scores were lower than were the second marker's, a fact which was due to two reasons. First, since the study was to show that creative responses were possible at the grade nine level, a more conservative score was preferable to a less conservative one. The experimenter therefore did not award marks to responses that were unclear or incorrect. The second marker accepted more responses as appropriate for the given situation. Second, the oral discussion revealed some ambiguous student responses to be statements of observation or of little depth. There is a danger of reading in meaning of greater depth into a response than is realized by the student himself. These sources of discrepancy are discussed and illustrated by specific examples in the following discussion.

The marker gave pupil I credit for a conjecture which was incorrect. At the time of scoring, the conjecture

> A series of pentagons can cover a flat surface without leaving gaps only if the flat surface is a pentagon.

was considered incorrect and unacceptable. The decision had been not to accept an incorrect conjecture unless it took careful analysis to judge its incorrectness. Because of the fact that many students realized that pentagons can not fit together without overlapping this hypothesis was not given a score. The marker disagreed that this hypothesis was inappropriate and thus the discrepancy of one in both the fluency and variety scores stood. (Table 11)

Similarly the discrepancy for pupil IV for conjecturing I is due to the following hypothesis.

> The lengths of the sides of the parallelogram can be altered to any extent as long as there are five sides.

The author felt that this hypothesis was unclear. The student, himself did not consider it any more than an obvious statement, as indicated by his oral comment that the pentagon could be an irregular pentagon.

The large discrepancy in the score for the second hypothesizing question for pupil IV was again because the author considered the hypothesis unclear, and thus inappropriate. The statements of discrepancy are as follows:

> i) the segments consisting of the base of the triangles would increase the same if all the triangles were expanded to a larger size.
>
> ii) No matter how the angles were expanded they would remain in the same proportion to one another.

The student's oral comments on the two ideas were as follows:

> I decided that if they had the same perimeter they have to have the same area and as the area increased in all of them they would have to remain the same size in proportion of one to the other.

From this it seems that pupil IV was expecting the triangles to grow as perimeter increased and to maintain the same shape under growth. When pressed for further explanation of the second statement, the student stated:

> If each triangle was blown up to say twenty times their normal size they would remain constant and in proportion one to the other.
>
> Would the angles have to remain the same?
>
> Yes. (The idea was stated very definitely)

As was indicated above, inappropriateness was decided on the basis of two ideas; incorrectness, or lack of clarity.

The judgment that any response was inappropriate may be questioned. In fact the idea of such a judgment being made may be questioned. However since mathematics is a science with definite guidelines some classification of responses was felt to be necessary. Taylor-Pearce established his ideas on subsets of concepts in an attempt to solve this problem. He also had a panel of mathematicians place some value judgment on the appropriateness of responses. Eastwood in his Trieste problem found that the same classification resulted to a 0.90 correlation by either an appropriateness scale or a judgment scale of established mathematicians. (Eastwood, 1965)

Certainly the author's decisions in a case such as that of pupil IV may be readily questioned. It is notable that IV, a high achiever (IQ-110) was so definite that the perimeter increase maintained shape. (If one thinks of them expanding one may think of them stretching equally in all directions.) In that case, pupil IV is awarded a much smaller score than he deserved.

The other hypotheses for pupil IV in question are the following:

i) All the triangles are found in space.

ii) They are all composed of dots side by side to form segments.

Pupil V also offered two hypotheses that provided a source for disagreement.

> If the perimeter increases, the area increases, but the number of degrees is the same. (180)
>
> All are scaled drawings, drawn to the same scale.

The first one was discarded because the first part of the hypothesis is a repetition of the original statement, the second, (the 180°) because it is a fact which was used in many problems during the experiment, and an idea which had been emphasized in the pretest and a fact which everyone had been exposed to in the classroom.

The marker, who had not worked with the students, suggested that maybe this was a thought which suddenly made sense to the girl. This indeed may be true and had happened many other times during the experiment. An idea, a concept which was presented many times in class suddenly was "discovered" by the student during his explorations. The role of the accommodation of these concepts is not to be underestimated (Dienes, 1960) and may very well have occurred here. The difficulty however is to maintain a consistent scoring scheme. Because this was such a common idea, many individuals, especially those more capable, more creative, would not even consider stating this fact. It thus seemed improper to give this response a score and thus to penalize those individuals who eliminated it, because of its obviousness.

Similarly the hypothesis dealing with the scale drawing seemed to be merely an observation from the grid. The marker suggested that it wasn't obvious that the lengths were drawn in proportion. A score given to this concept also seemed to be penalizing those individuals who were attempting to establish more general relationships.

A statistical correlation of reliability was not obtained since only one other person marked a small sample of the tests marked by the author. A second individual had been asked to mark a second sample; a statistical report would then have compared the scoring for the three samples; however the time required for adequate scoring was too great and the scores were not awarded. Because of this, with the cooperation from the first scorer, the detailed description of the scoring discrepancies was presented instead.

SUMMARY

In the present chapter, a scoring procedure for each problem was established. The conjecturing and the sensitivity problems were scored for fluency, variety and novelty. A hierarchial type of scoring scheme resulted for Sensitivity II. This necessitated some modification, that of adding the number of responses in a category that was a subset of the specific category involved to the number of responses in the category itself, in order to obtain the number of total responses in that category.

The responses to Redefinition IIA, and Redefinition IIB were assigned scores of one or two, depending upon the efficiency of that response. Redefinition I was given a score which reflected the length of time taken for solution. The responses for verifying were classified into categories that were thought to reflect a level of sophistication in verifying. These categories were assigned ranks of zero to six.

The question of reliability was treated in a very intuitive and descriptive sense. An attempt was made to describe areas of and reasons for discrepancies found between the scores of one marker and that of another.

CHAPTER VI

RESULTS

Chapter VI has been divided into two sections. The first deals with the results from the image analysis performed on the correlations between the test scores, the second with the direct comparison of correlations between pairs of scores. Although some of the conclusions drawn from the image analysis are dependent upon the discussion based on examining the correlations between pairs of test scores, it was felt that the discussion on the image analysis would provide the reader with an overall view of the nature and composition of the problems and therefore a point of reference for the later discussion. The reader may also wish to refer to the constructed problem-situations during the discussion. For convenience, they are presented again in Appendix C.

IMAGE ANALYSIS RESULTS

Do the test situations constructed to measure the processes defined by the model (Chapter III) actually represent these processes? In this chapter the correlations between pairs of scores on the problems, on the SCAT tests, the achievement test, and the Torrance tests are compared in an attempt to establish the constructs measured by the constructed problem-situations. The image analysis was performed in order to establish whether four factors, representative of the four processes, did exist in the constructed tests. If the conjecturing problems were all found to load highly on one factor and not to load on any other factor, such results would support the assertion that these tests had in common and were representative of this one ability or process. The conjecturing process could then be further described by examining the intercorrelations present among the tests, the content in each problem, and the oral discussions of the pupils working on the problems. In summary, the image analysis results are to be used to help describe the influences present in each test, to help substantiate that one process question is indeed like another, and to help isolate at least four processes by which children solve problems.

A total of twenty-six scores were included in the image analysis--the twenty-one scores from the constructed problems, three Torrance scores, and two SCAT scores. All the scores for each constructed problem were included. A varimax rotation was used in an attempt to define independent dimensions that could be interpreted in terms of the model. The image analysis was done three times, the first calling for ten factors, the second for six factors, and the third, for five factors. Six factors were indicated by the theory and the correlational results but the five-factor and ten-factor analysis were also obtained in an attempt to see if more adequate interpretation was possible from these analyses. After examination, the six-factor solution was considered most appropriate for the reasons discussed below.

First, the Scree test on eingenvalues greater than one was applied. (Hakstian, Bay, 1972, p. 24-26) The break after the sixth eignevalue (Figure 1) seemed to support the hypothesis suggested by the theory that called for four processes, the ability tests, and the Torrance tests as the six factors. Table 12 presents the loadings on the six rotated factors extracted from twenty-six test scores obtained for the sample in this study. The discussion included in this chapter is based on the results presented in this table.

The results from the five-factor analysis did not promise more information than did those from the six-factor analysis. In fact, the scores separated into clusters that seemed analogous to those in the six-factor solution. The results from the five-factor analysis are reported briefly in Appendix B.

Similarly the loadings on the ten-factor analysis did not seem to add further interpretability to the variables. The six main clusters were again indicated and the remaining four factors were identified by one or two separate scores. As a result the problem of identifying the factors did not seem to be simplified and therefore this result was not reported.

The Torrance tests were found to load heavily on Factor IV. No other variable from the twenty-six included in the analysis loaded on this factor. Since this distinct and separate loading on a single factor also occurred in the five-factor analysis, it might be assumed that little or no relationship exists between the constructed tests and the Torrance tests. Apparently, the children

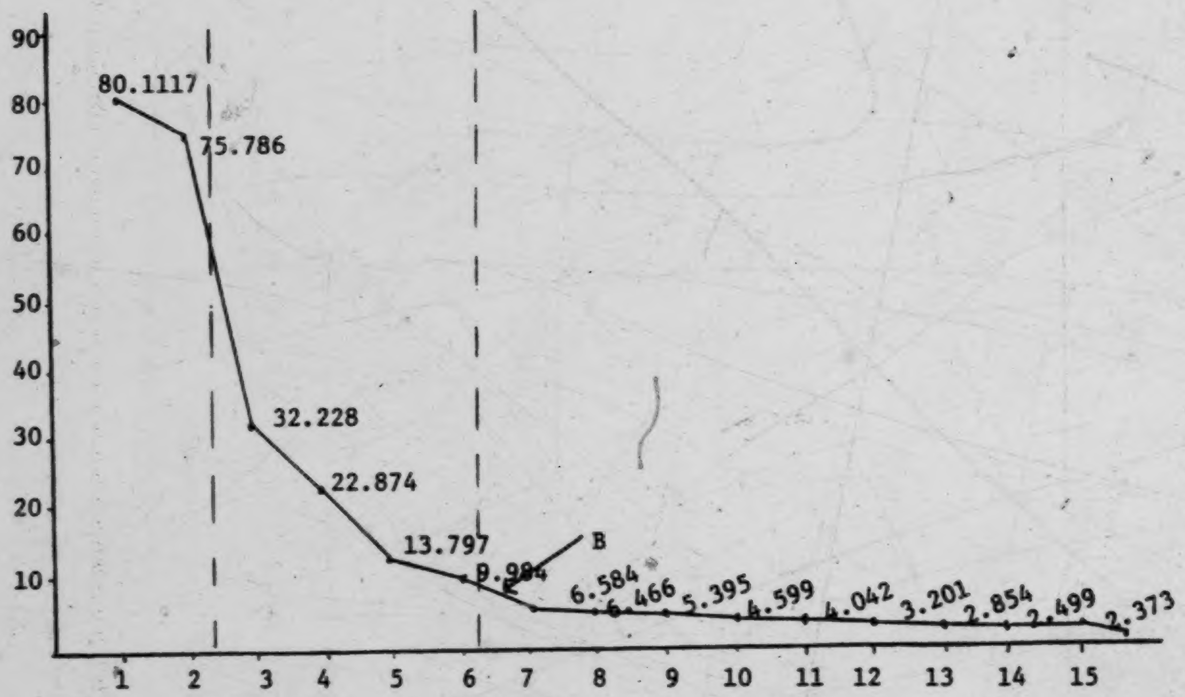


Figure 1

Eigenvalues for the First Fifteen Image Factors Extracted from Twenty-Six Test Scores Representing Creativity, Ability, and the Constructed Problems

TABLE 12

SIX IMAGE FACTORS EXTRACTED FROM TWENTY-SIX TEST SCORES REPRESENTING CREATIVITY, ABILITY AND THE CONSTRUCTED PROBLEMS

VARIABLE	1	2	3	4	5	6	COMMUNALITIES
SCAT Verbal	-	.36	.43	-	.35	-	.47
SCAT Quantitative	-	-	.69	-	-	-	.69
Torrance F	-	-	-	.96	-	-	.94
Torrance V	-	-	-	.90	-	-	.85
Torrance N	-	-	-	.97	-	-	.96
Conjecturing IF	-.30	.63	-	-	-	-	.56
Conjecturing IV	-	.77	-	-	-	-	.65
Conjecturing IN	-	.80	-	-	-	-	.71
Conjecturing IIF	.30	.62	-	-	-	-	.55
Conjecturing IIV	.45	.63	-	-	-	-	.73
Conjecturing IIN	.30	.50	-	-	-	.47	.65
Sensitivity IF	-	-	-	-	-	.71	.52
Sensitivity IV	-	-	-	-	-	.86	.84
Sensitivity IN	-	-	-	-	-	.69	.59
Sensitivity IIF	-	-	.47	-	.81	-	.90
Sensitivity IIV	-	-	-	-	.95	-	.95
Sensitivity IIN	-	-	-	-	.86	-	.76
Redefinition I	.32	-	.70	-	-	-	.61
Redefinition IIA	.39	-	.61	-	-	-	.62
Redefinition IIB	-	-	.46	-	-	-	.30
Verifying Ia	.84	-	-	-	-	-	.83
Verifying Ib	.64	-	-	-	-	-	.45
Verifying Ic	.74	-	.33	-	-	-	.69
Verifying Id	.79	-	-	-	-	-	.80
Verifying IIA	.36	-	.50	-	-	-	.45
Verifying IIB	-	-	.52	-	.32	.38	.54
TOTAL COMMUNALITY	3.15	3.00	2.96	2.92	2.80	2.51	17.61
% COMMUNALITY	18.5	17.1	16.8	16.7	15.8	14.2	

*The Varimax Rotation of Factors was used.
*Only Loadings greater than .30 are recorded.

did not treat any of the constructed questions in the same way that they treated the Torrance tests.

This result was surprising. The Evans tests, used as exemplars for the open-ended situations, correlated with the Torrance tests. (Evans, 1964, pp. 193-194) Many of Torrance's ideas had acted as stimuli for the rationale behind the study. Nevertheless, the separation of the Torrance tests and the constructed problems, to the extent that the latter represent the processes hypothesized by the model, does not necessarily invalidate the model. Creativity has been shown to be a composite entity, described by multiple characteristics. (Chapter II) For example, Guilford's (1967) factors for creativity were drawn from orthogonal components defined by his model, The Structure of the Intellect. It may be that the Torrance tests describe one aspect of creativity, and that the constructed problems represent another aspect or aspects which are orthogonal to that viewed by Torrance.

The statistical indication that the children responded in a different way to the Torrance tests than they did to the constructed problems was supported by the experimenter's observations. The Torrance tests appeared to be treated in one of two ways--as playful or as threatening. The students from one school appeared to treat them very playfully and lightheartedly. They did not seem to treat the questions with any of the seriousness or intent they gave to the constructed questions. Students from the fourth school, however, appeared to resent and to feel threatened by the total experience and their reactions seemed to reach a peak when they were confronted with the Torrance tests.

The students of the second school, who had reacted very favorably to the experiment, who had enjoyed the constructed questions and had actively participated and cooperated in the experimenter's attempts to examine their thinking patterns, seemed to think the Torrance tests almost irrelevant and too silly to bother with. The students in the third school did not appear to feel very strongly about the Torrance tests but indicated that they found these less informative and less enjoyable than the constructed problem-situations. Perhaps the students saw all the constructed questions as having more direction and, having a more established expectation on the appropriateness of solutions, were more comfortable with them.

The Torrance tests and the constructed problem-situations also differ in the specificity of the content describing the tests. Torrance's verbal test reminds one of creative composition. It seems more related to the field of story-telling, rather than to mathematics. The figural test requires the individual to complete a series of given line segments in as many ways as possible. It rewards the individual who can spatially conceptualize the individual pieces as part of a greater whole. Both the verbal and the non-verbal forms were scored for fluency, flexibility and originality. These scores would not necessarily reward the individual who built one idea on a previous one in the constructed problems. Many of the responses to the mathematical questions were ideas which interrelated; one idea often lead to another. (Specific responses to the constructed problems will be discussed further in Chapter VII)

Factor II is characterized by the high loadings (0.50-0.80) of the two conjecturing problems. The SCAT verbal also contributes a loading of 0.36 to this factor, implying an association between verbal ability, The Pentagon, and The Triangles. This association may be due to several factors. First, both conjecturing questions require formulation of hypotheses, which may be generated by a verbal manipulation of the characteristics of The Pentagon (or The Triangles) so as to obtain a sequence of possible relationships. This isolation of the verbal form of the variable, followed by the proposal of several combinations of this variable with verbal forms of other variables, may have occurred with little intuitive understanding of the mathematical concepts behind the variables. (Chapter VII, page 250) Second, the verbal SCAT scores correlate highly with intelligence and it is possible that the common part shared by the SCAT verbal and the conjecturing questions may be due to the influence of intelligence. This suggestion, however, is not supported by the correlations between scores on the conjecturing situations and the scores on the Lorge-Thorndike tests which are not significantly different from zero. (See Table 15, page 185)

A third possibility is that the relationship between the verbal comprehension test, vocabulary test, and the conjecturing problems is due to general knowledge. Individuals who read more extensively may tend to score higher on the vocabulary tests. This superior general knowledge may also enable the student to react more capably to tests which call upon the student to formulate his own conjectures.

The SCAT Quantitative test measures some numerical facility of the type required by questions such as the following:

i) Change .4375 to a fraction.
ii) A hotel needs window curtains requiring 1 1/8 yards of material each. How many of these curtains can be made from a 72 yard bolt of material?

Since both the SCAT Quantitative test and the conjecturing situations are mathematical, a relationship between these scores may have been expected. This did not seem to occur and the SCAT quantitative scores did not load on Factor II. This lack of relationship between the quantitative ability test and The Pentagon (or The Triangles) would indicate that thinking required to answer questions of the type illustrated by the two items from the SCAT does not appear to relate to the type of thinking required to formulate hypotheses about a pentagon or about a series of triangles.

Some students appeared cognizant of the presence of the conjecturing type of question, at least as an open-ended situation. As stated by one girl who was asked to comment on the experiment: "It seems that there were two types of questions, some for which there were many possibilities, others for which there was only one answer." (This girl preferred the open-ended situations.)

In summary, Factor II seems to be mainly characterized by The Pentagon and The Triangles and thus it is postulated that this factor is representative of the conjecturing process. Since the loadings for Conjecturing I, The Pentagon, Variety, (0.77) and Novelty, (0.80) are the highest of the six loadings on Factor II, these scores will be considered as the best indicators of the conjecturing process.

The verifying and the redefinition problems loaded almost entirely on Factor I and on Factor III. Factor III is characterized by high loadings by the scores for Redefinition I, The Board and the Hole (0.70), Redefinition II, Angles (0.61), and Areas (0.46), as well as scores for both SCAT tests (0.43, 0.69). Since the SCAT tests measure "school learned abilities directly rather than psychological characteristics. . .which afford indirect measurement of capacity for school learning" (Buros, Sixth Yearbook, p. 717), it seems that Factor III involves processes that are emphasized in the traditional classroom situation. This result would seem to imply that those students who performed well on the three redefinition questions also perform well in the traditional classroom. This same conclusion would also seem to apply to Verifying II, Parallel Lines.

What is the nature of the trait shared by Redefinition I (The Board and the Hole), Redefinition II (Angles and Areas), Verifying II (Parallel Lines), Sensitivity II (Area Increase) Fluency, and the SCAT tests? The tests of redefinition may be tests of scholastic ability. Redefinition II--Angles and Areas--require responses which are closely related to traditional classroom achievement. The emphasis on achieving a solution by the most efficient way might tend to increase the relationship between the measures of ability and the redefinition questions. Similarly, Verifying II, Parallel Lines, calls upon the student to use the knowledge about angle relationships, given the condition of parallelism, that he acquired in the classroom. This is especially true for Verifying II, part B, which required the student to find the measures of given angles.

Verifying IIB shares a further characteristic with the redefinition questions. Both require answers with little or no theoretical justification. The numerical data included in Parallel Lines, part B, enables the student to solve the problem without formal deliberation on the reason for equal angles. Area Increase, scored for fluency, may also emphasize the close relationship to classroom work and to including various examples of length and width increase without any justification of the reasoning process required to see the influence on area.

Further, the situations loading heavily on this factor may be a reflection of an individual's ability to consider several variables at one time. The Board and the Hole clearly illustrates the need to distinguish and to manipulate two different variables. The individual is required to manipulate the idea of one-half with the number relationships relating the physical dimensions of width and length. Sensitivity II, Area-Increase requires the separation of length and width and the comparison of their increase in various combinations to the area increase. It also requires the discarding of the supposed requirement that the rectangle remain a rectangle under the side increase. On the other hand, it is not clear that this ability to consider several variables in a multiple-step solution is necessary for the second redefinition situation. However the association may be due to the fact that many students attempted to find the area measures and the angle measures by analyzing parts of the given information, then in combining them.

In conclusion, Factor III appears most related to the processes stressed by the school ability tests, most notably by the quantitative ability test. (The loading for the SCAT numerical score was 0.69.) This process appears to involve an ability to reorganize data to meet new requirements. (For example, the dimensions of a board of area of thirty square units have to be altered to new specifications while retaining the same area.) This seems to be the same process that is required to solve problems about the cost of curtains, or about ratios, that is, traditional mathematical problems which required more than one reasoning step in the solution.

In summary, the situations describing Factor III had three common elements: i) the reorganization of data to find an interpretation different from that usually acceptable, ii) the solution of a multiple-step problem, and iii) the use of content and thinking patterns similar to those emphasized in the classroom. This ability to realize a certain end and to use variables from the problems in various combinations in order to achieve the perceived requirement meets the original definition for the hypothesized process of redefinition: i) the act of reassociating and recombining previously unassociated elements of knowledge to result in new combinations, and ii) the act of discarding of a previously consistent approach in order to facilitate the perception and the solution to a problem. The number of combinations and the extent to which one combination was suggested by a previous one differed from one individual to the next. The numerical analysis cannot help us know how well the individuals discarded or how quickly they changed their approach to a problem. A closer investigation into the methods used by the students is needed to give insight into this question.

Factor I was largely described by the high loadings (0.64-0.84) of the scores for Verifying I, The Polygons. Verifying IIA, the abstract part of Parallel Lines, also loaded (0.36) on this factor. These situations all required that the individual justify to the best of his ability the statement given in the problem. These justifications were made by responses that may be considered to approximate the definition of verifying presented in Chapter III: i) testing by use of specific examples, ii) finding of a justification or a rationale of assumptions, iii) proving a statement by logic, or deduction, iv) producing suggestions by which the subject may test the statement.

The second conjecturing problem, The Triangles also loaded (0.30, 0.45, 0.30) on this factor. This may be due to several characteristics of this problem. First, the question presented to students may require different processes from an individual depending upon that individual's level of experience with the concepts involved in the question. Conjecturing II, The Triangles, seemed to have two levels: the first, a verifying plateau from which the problem-solver could then extend into the second, the conjecturing, aspect. Conjecturing II, presents a series of triangles of equal perimeter and different area. The sample conjecture (incorrect in that it was incomplete) suggests that the area for a triangle increases as its perimeter increases. Most of the students did not seem to have full understanding of the relationships involved and therefore became very interested in examining the reason why area did not remain constant when the perimeter had done so. For these individuals the question then took on some of the nature of a verifying question. They attempted to construct more triangles of perimeter equal to the given triangles and proceeded to calculate the area of these triangles (usually by the area formula) in order to establish the specific relationship between area and perimeter to their own satisfaction. Just as these students had generated specific instances in the verifying problems, so they tried to generate specific instances for the series of triangles in Conjecturing II. In this respect, the question lost some of its open-endedness, becoming more limited than the pentagon question, but not quite as limited as the side-area question involved in Verifying I, since there were at least three obvious possibilities to consider:

i) If the perimeter increases, the area increases.

ii) If the perimeter is constant, the area increases just by changing the altitude base relationship.

iii) If the perimeter increases the area may remain constant.

The process of generating or working with specific instances often did set the scene for the question, "Why?", and eventually for a conjecturing situation, as seen by the high correlations between scores on Conjecturing II and Conjecturing I (Table 13, page 177).

An interaction between the verifying and the conjecturing processes may be necessary and fruitful in achieving solutions to problems. Such interaction as observed for The Triangles emphasizes the difficulty of constructing problems which reflect only one process, particularly when these problems are designed for students of heterogenous and unsophisticated mathematical experience.

Second, the association between Conjecturing I and Verifying I may be due to their common content of area-perimeter in certain polygons and the effect of this relationship on angle size.

Redefinition I, The Board and the Hole, and Redefinition IIA, Angles, also load on Factor I (0.32, 0.39 respectively). If Factor I is to be described as representing the process of verifying, then some aspect of Redefinition I and Redefinition IIA should be interpretable in terms of this process. One approach to the solution of The Board and the Hole was a trial and error method which could be similar to the testing of specific suggestions. It is also possible that the concepts of area (constant at thirty square units) and perimeter (varying from 10 x 3 to 15 x 2) may have resulted in the association of this problem with the conjecturing situations and with Verifying I.

The loading of Redefinition IIA is difficult to interpret in terms of either process or content held in common with the other situations that load on Factor I. This problem is described in greater detail later in the chapter (see page 212).

The two sensitivity problem-situations did not load on the same factor. This reflected the lack of correlation between scores on the two problems as reported in Table 16 (page 189). The scores on Verifying IIB, the specific part of Parallel Lines, and Conjecturing II, Novelty, loaded (0.38, 0.47 respectively) on Factor VI which is characterized by Sensitivity I, The Square. The SCAT verbal scores and the Verifying IIB scores loaded (0.35, 0.32) on Factor V characterized by Sensitivity II, Area-Increase.

The first sensitivity problem, The Square, theoretically appears to require two types of thinking. First, the realization that a straight cut is unnecessary has to be made and second, some sort of hypothesizing as to the cuts possible, followed by a classification of the types, should be formulated. Only one individual progressed to a full classification of types. A second person generated several types and then produced a very elaborate artistic division. Statistically these two extremes were considered equivalently creative.

Factor VI, characterized by the Square (0.69-0.86), may involve the manipulation of ideas in response to an unrealized and unspecified goal. In comparison, for the process redefinition, as reflected by Factor III, the goal was established; manipulation was goal-directed and the goal was very specific. For The Square, the goal was not specific although a definite direction for response was indicated. This possible description of Factor VI is also supported by the 0.47 loading of Conjecturing II (Novelty) on this factor. Those responses to Conjecturing II (The Triangles) regarded as most novel extended beyond the concepts of area and perimeter suggested by both the given conjecture and the usual school activities about triangles. These were manipulations of specific ideas about measures of triangles in an attempt to realize some new ideas about the given set of triangles in the situation.

The second sensitivity situation seems to have a verbal component inherent in it, and this may be the reason for the high loading on Factor V (0.81-0.95), a factor which also involves the SCAT Verbal scores (0.35). The ability to delve into the meaning of "two sides" from a verbal point of view, which seemed to influence a correct solution as indicated by several students, appears to be closely related to general intelligence. The solution of the problem required individuals to break the "set" of expecting a rectangle to remain a rectangle [illegible] change, but also required the respondent to inquire as to the meaning behind the words used. This step often did not occur in the observed performance; "two sides" automatically meant "parallel and opposite sides" to the students.

The ability to question the meaning behind "two sides" did not always imply that the "set" of the rectangle would be broken. During the oral interview many individuals were able to interpret the meaning for two sides verbally and yet were consistently unable to draw a figure which met the specifications of the problem and yet was not a rectangle. Just which of the two above ideas (the verbal influence, or the ability to break the set of the rectangle) is most reflected by Factor V is not known. Neither the five-factor, nor the six-factor analysis seemed to help in characterising the process. Certainly if it is the set-breaking ability that is descriptive of Factor V, then it does not seem to relate to the breaking of the geometrical set of one-half required by The Square and thus Factor VI.

Both possibilities for Factor V (the verbal component, and the ability to break the set of the rectangle) can be justified by the consistent loading of the specific second verifying situation (Verifying IIB) on this factor. There appeared to be a need to break a set toward the special case of equilateral triangles in Parallel Lines. (See page 146) There also seemed to be a language problem with the second part of Verifying II, Parallel Lines. Some individuals indicated that the second part of Verifying II was more difficult than the first part because of the mathematical wording used in outlining the question. This consideration may be the one which was most influential in determining responses to the question.

How do these descriptions relate the two sensitivity problems to the process of sensitivity as defined by the model? The definition of sensitivity was specified as follows: i) the ability to perceive deficiencies and errors, shortcomings or inadequacies in a given situation, and ii) the ability to see possibilities in a given situation, possibilities that lead to further questions. The ability to formulate ideas on the cutting of the square in half (Sensitivity I) appears to meet both criteria. First, the student must critically assess the idea of "one-half" and straight-line cuts and secondly, he must be able to produce some examples of these cuts. The ability to question the meaning of "two sides" (Sensitivity II) reflects the critical attitude inferred by the first criteria. The ability to draw an alternative figure to the rectangle which meets the specification for the increase in the lengths of two sides (Sensitivity II) reflects the second criterion in that the individual must be able to produce the possibilities present in the situation.

It is possible that the ability to see shortcomings in a problem may not necessarily be dependent on the ability to see possibilities that lead to further exploration. These may be two independent facets in the process of being sensitive. This generalization reflects the findings reported by Rossman (1931) and by Torrance (1965) that a critical attitude is not necessarily equivalent to an inventive attitude. (See detailed report, Chapter III, page 56)

Since none of the students were able to produce the range of alternative possibilities (see Chapter V, page 135) for the rectangle in Area-Increase, this problem may reflect only the critical level of sensitivity. This may explain in part why the scores for this question loaded on a factor different from that loaded by the scores for The Square, which reflected both aspects of sensitivity. There were other aspects present in Area-Increase which may explain the separation of scores for this problem from that for The Square. These are examined in light of the correlations between pairs of scores later in this chapter. At this point in the discussion, however, it may be suggested that the process of sensitivity may be described by two separate factors. Later in the chapter, Factor VI is described as the superior representation of the process of sensitivity.

The image analysis identified six clusters of scores which could be interpreted in terms of the hypothesized model. Five of these clusters were characterized by loadings by scores from individual constructed problem-situations. Factor IV was identified very clearly as the creativity defined by the Torrance tests. This aspect of creativity does not seem to be the same process as that reflected by any of the other tests used in the analysis. Factor II seems to be a clear representation of the conjecturing process. Factor I was not as clearly interpretable but seems to be consistent with the discussion on verifying. Factor III seems to represent the process of redefinition and seems to associate very closely with abilities emphasized by traditional school activities in mathematics. The process of sensitivity cannot be clearly identified from the image analysis. Either Factor V or Factor VI may represent this process. Factor V is identified by Area-Increase and seems to include a redefinitive and a verbal component. Factor VI is characterized by The Square, and seems to include redefinitive and conjecturing aspects. Further description of these factors occurs later in the chapter in terms of the two questions and their correlations to the other variables in the study.

THE CORRELATIONAL RESULTS

The following discussion, based on the correlations between pairs of scores provides further description of the questions and their responses. The discussion is presented in four sections, one section for each process. Each section is organized about the following questions.

i) Do the scores for the two situations on each process (conjecturing, sensitivity, redefinition, verifying) correlate? Do the scores between different measures on the same problem correlate?

ii) Do the scores for the situations for each process (conjecturing, sensitivity, redefinition, verifying) correlate with the scores for the other constructed situations? This discussion will investigate the extent to which the problem-situations designed for one process measure the same construct as those designed for the other processes.

iii) Do the scores for the constructed problem-situations correlate with the scores on the standard measures?

Analysis of the Process of Conjecturing

How well do the two conjecturing problems relate? To what extent do Conjecturing I, The Pentagon, and Conjecturing II, The Triangles, involve the same process? Table 13 presents the correlation between pairs of scores on the two problems.

The correlations between measures on the same situation were found to be significantly different from zero; the smallest correlations present between the novelty and the fluency scores in each case. That the correlations between the fluency scores and the novelty scores for each problem are smaller than the correlations between fluency and variety or between variety and novelty would be

TABLE 13

INTERCORRELATIONS AMONG THE FLUENCY, VARIETY AND NOVELTY SCORES FOR THE TWO CONJECTURING PROBLEMS

	Con. I Fluency	Con. I Variety	Con. I Novelty	Con. II Fluency	Con. II Variety	Con. II Novelty
Con. I Fluency	1.00	-	-	-	-	-
Con. I Variety	.77**	1.00	-	-	-	-
Con. I Novelty	.38*	.55**	1.00	-	-	-
Con. II Fluency	.26	.34*	.51**	1.00	-	-
Con. II Variety	.19	.32*	.57**	.66**	1.00	-
Con. II Novelty	.05	.18	.54**	.54**	.55**	1.00

* $p < 0.05$
** $p < 0.01$

expected. For fluency, all responses were accepted, for novelty a classification and subjective judgment was applied.

The correlation between fluency, Conjecturing I, and the scores on Conjecturing II were not significant, however the novelty scores of Conjecturing I did correlate highly ($r = 0.51, 0.57, 0.54$) with the scores on Conjecturing II. This may have been due to the fact that students found it easier to evaluate their own responses for The Triangles than for The Pentagon. Most of the responses to The Pentagon that were not observations were obtained by adding some further limitation or by varying some specification of the original hypotheses, or an observation about the triangles in the pentagon. (See Chapter VII, page 225) Eighty-seven percent of the responses for The Pentagon fell into the categories deletion of data, addition of data, or variation of specifics whereas fifty-six percent of the responses for The Triangles were classified into the same three categories. The greater spread in responses for The Triangles could result in the variety and novelty scores more closely approximating the fluency scores for this question.

The high interrelationship between the three parts of each conjecturing question would suggest that one of the scores could be a valid representation of an individual's ability to perform on this task. Since the novelty scores between The Pentagon and The Triangles seem to correlate most consistently, the novelty score may be the best representation of an individual's ability to conjecture.

There is, however, some advantage to using the variety score as a measure of conjecturing. Although the correlations between the two variety scores (r = 0.32) are lower than the correlations between the two novelty scores (r = 0.5[illegible]), the variety score is more objective than the novelty score. Obtaining the novelty score required organizing the responses into categories of "equivalent statements". The equivalence of such statements was decided upon by the experimenter and not only included statements which said the same in different words but also two statements which seemed equivalent in value. For example, the division of the pentagon by its diagonals might result in the descriptive statements:

i) A pentagon is made from a star and five triangles.
ii) A pentagon is made from three triangles.
iii) You can make five triangles by joining the vertex and the sides.

The descriptive statement resulting from joining the vertices and the center point is considered to be "equivalent" to the statement resulting from joining the diagonals. These statements were judged by the experimenter to be alike in the amount of thinking and manipulation to the figure which gave rise to the information described in the statement. (Chapter V, page 125ff) This type of categorization requires some interpretation of the possible depth and implication of the child's hypotheses. In categorizing the two statements into the same class, there is a judgment of the possible value of the response. This bias does not occur in the "variety" scoring procedure.

Table 14 and the discussion following the table presents information on the extent to which the conjecturing problems measure the same constructs as do the other constructed problem situations.

The first conjecturing situation, The Pentagon, does not

TABLE 14

CORRELATIONS BETWEEN PAIRS OF SCORES ON CONJECTURING AND THE OTHER CONSTRUCTED TESTS

TEST	SEN. IF	SEN. IV	SEN. IN	SEN. IIF	SEN. IIV	SEN. IIN	RED. I	RED. IIA	RED. IIIB	VER. IA	VER. IB	VER. IC	VER. ID	VER. IIA	VER. IIB
Con. IF	-	n	-	n	n	n	n	n	n	-.4*	n	n	n	n	n
Con. IV	-	n	-	n	n	n	-	n	n	n	n	n	n	-	-
Con. IN	-	.32*	.42**	-	-	-	-	-	-	n	n	n	-	-	-
Con. IIF	-	-	-	-	-	-	-	.35*	-	-	-	-	-	-	-
Con. IIV	-	.41*	-	-	-	-	.42**	.42*	-	.35*	.48*	.31*	.46**	.43**	-
Con. IIN	.35*	.47**	.44**	-	.36*	-	-	.40**	.31*	.34*	-	-	.39*	-	-

* $p < 0.05$
** $p < 0.01$
n indicates a negative correlation

Legend:
F--Fluency
V--Variety
N--Novelty

correlate with the sensitivity situations, the redefinition situations, or the verifying situations. The only exception occurs between the novelty score on The Pentagon and the novelty (r = 0.42) and the variety (r = 0.32) scores for The Square, Sensitivity I. This may be a result of the type of scoring procedures used for both situations. In both cases it was necessary to categorize the responses into groups of equivalent responses. The process of categorization, the process of judgment imposed by the scorer may have resulted in the similarity in "appropriateness" standards.

The significant correlations between these scores may be due to a second reason. The quality of response for The Pentagon often depended upon the relationships observed by the student as he divided the pentagon into various parts. This spatial insight was also required in finding new ways to cut a square in half.

The second conjecturing situation, however, interrelated clearly with the other constructed problem-situations. There is a clear relationship between the novelty scores of The Triangles and the variety scores of both the sensitivity questions (r = 0.47, 0.36). This again may reflect the scoring procedure. However, the fairly high relationships (r = 0.31, 0.42) between The Triangles and the two redefinition situations indicate that there may exist a level of predetermined set in The Triangles, a characteristic which is shared by Sensitivity I, The Square. In The Square this set was intentional; the students have to be sensitive to the existence of numerous ways to cut a square in half before they are free to conjecture these numerous ways. The need to overcome a set, however, had not been an intended characteristic for The Triangles, but may have occurred because of the expectations developed from the school curriculum which is mainly concerned with area and perimeter relationships for The Triangles. For this problem, it is necessary to break from the area-formula treatment for individual triangles and to look at the set of triangles as a unit or pattern which can be extended before new conjectures can be formulated. Many children had an easier time extending the pentagon relationships, perhaps because they did not have a predetermined standard of what was correct or what was expected. The existence of a predetermined expectation of what constitutes a good answer may also explain the relationship between The Triangles and Redefinition II--Angles. The content of the latter question is very closely related to that traditionally found in the classroom tests.

The redefinitive aspect of The Triangles is emphasized by the correlations of the novelty score with both parts of Redefinition II ($r = 0.40$ and 0.31) and of the variety score to The Board and Hole question ($r = 0.42$). Why did the variety score correlate with The Board and the Hole and the novelty score correlate with Redefinition II--Areas? A procedure based on time was used to score The Board and the Hole. The individual who scored higher on The Board and the Hole was capable of being more flexible within a shorter time. This individual would tend to produce more responses of various types (variety) within a specified time than the individual who required a longer time to produce the alternatives necessary to solve The Board and Hole problem. The scoring for

Redefinition II--Areas rewarded the rearrangement of parts of figures into more regular polygons as a means of finding area. Fewer individuals used this approach and most of the solutions were found by division of the polygon into smaller areas. This occurrence may have resulted in the correlation between this score and that of the novelty score for Triangles, which was also a statistical score.

The correlation between the verifying situations and The Triangles ($0.31 \leq r \leq 0.48$) may be due to two factors. The content in The Polygons was that of the area and perimeter of the shapes specified. The treatment of triangles in the classroom curriculum emphasized the area-perimeter setting and therefore the response to Conjecturing II, The Triangles,may reflect this aspect of the students' knowledge. Many students responded to Conjecturing II, The Triangles, at a verifying level. To them the area-perimeter relationships appeared to be insecure and they had to test out ideas such as "if the perimeter increases, the area also increases."

The lack of correlation between Conjecturing II, The Triangles, novelty, and the second and third parts of Verifying I, The Polygons, may be in part attributed to the large number of people who scored zero on the verifying questions. Responses were made, but many of these were only descriptive statements. However, this observation does not explain the significant correlation between part d and The Triangles, since many zeros were also scored on part d. The irregular pattern present in the correlations between the verifying situations and the second conjecturing situation may be due to some irregular pattern of responses given to the verifying questions. (See page 207ff)

Table 15 presents the correlations between scores on the conjecturing situations and the scores on the standard measures of creativity, ability and achievement. The discussion about these correlations will focus on the extent to which the constructed measures reflect abilities measureable by the already existent tests.

The correlations between the conjecturing situations and the Torrance tests indicate that the experimental questions measure something different from what is measured by the Torrance tests. Some aspects of this surprising result have been discussed on pages 158 and 161 of the former section in this chapter.

The suggestion at that time was that the constructed tests and the Torrance tests measured two independent facets of the multifaceted ability, creativity. The following two observations may summarize the reasons for considering that the two situations measure different processes:

i) The content of the experimental problems and the Torrance tests differs. The situations constructed for this experiment are specific to mathematics. They deal with content which is applicable to the mathematical learning situation present in the school curriculum today. The Torrance tests are nonspecific, possibly being closer to language arts than to any other field. It may be possible that many of the children's answers to the Torrance items are influenced by their knowledge from television and science fiction. The figural tests which call upon individuals to draw

TABLE 15

CORRELATIONS BETWEEN PAIRS OF SCORES ON THE CONJECTURING PROBLEMS AND THE STANDARD MEASURES OF ABILITY, ACHIEVEMENT AND CREATIVITY

	SCAT Q	SCAT V	MATH ACH.	NVCR. F.	NVCR. FLEX.	NVCR. ORIG.	NVCR. VER. ELAB.	VER. F.	VER. FLEX.	VER. ORIG.	LORGE THORNDIKE V.	LORGE THORNDIKE N.V.
Con. IF	n	n	n	-	-	n	-	-	-	-	n	n
Con. IV	n	-	n	-	-	n	-	-	-	-	n	n
Con. IN	-	.33*	-	n	n	n	-	-	-	-	n	-
Con. IIF	.41**	.35*	.45**	n	n	n	n	n	-	n	-	n
Con. IIV	-	.42**	-	n	n	n	-	-	-	-	-	-
Con. IIN	.35*	.53**	.46**	n	n	n	n	n	n	n	.31	n

* $p < 0.05$
** $p < 0.01$
n indicates a negative but insignificant correlation

Legend: F--Fluency
V--Variety
N--Novelty
NVCR.--nonverbal creativity score
Ver.--verbal creativity score

pictures suggested to them by couplets of straight lines or other similar basic figures, do not seem to be specific to any field of study.

ii) The process involved in the Torrance tests and the constructed situations may differ. The mathematical tests measure an individual's ability to work in a limited situation and in answer to a specific standard. This requirement is certainly not the same as that of trying to suppose consequences to putting strings on clouds. (Torrance, 1966) The former task places before the student a definite expectation, the latter a fanciful and playful situation which seemed to be treated in a more light-hearted manner by the students. The results would not necessarily be expected to relate, because of possible differences in emotional and mental set for the two tasks.

Conjecturing I, The Pentagon, did not correlate with the achievement or the ability tests except for the relatively low but significant correlation ($r = 0.33$) between the novelty score for The Pentagon and the SCAT verbal ability test. On the other hand, Conjecturing II, The Triangles, correlated with both the ability and the achievement tests ($0.35 \leq r \leq 0.53$). The Triangles seemed to relate to questions which were not open-ended, that required numerical and mathematical knowledge used in the typical classroom work. It was noted earlier that students at a certain level seemed to respond to this question as they had to verifying situations. This type of response may account for a significant part of the correlation between achievement and The Triangles. (Verifying also correlated with achievement)

Summary.

First, the scores of the problems for the different measures of fluency, variety, and novelty seemed consistent with each other. It may be concluded that one of the three scores for each question may represent the performance on that question. The novelty score is cited as the score providing the best statistical association between the two problems, however the variety score is claimed to be the more objective of the two. The fluency score is considered least appropriate of the three because it is only a measure of the number of responses rather than of the quality of responses.

Second, the pentagon problem seems to involve a fairly distinct process. The Pentagon meets the specification assigned for conjecturing, that is, students interpreted the problem as an open-ended situation and were able to respond to the situation at various levels. Factor I, identified by the loadings by The Pentagon, may thus be interpreted as a conjecturing factor. The Triangles, seems to have at least two distinct plateaus. One may be described as a verifying plateau at which the students justified the suggested relationship. The second is the conjecturing plateau called for by the question, a third level may be distinguished as the verifying that occurs in an interlocking fashion with the conjecturing that is done. Nevertheless the two conjecturing problems correlated at a level significantly different from zero and therefore may be considered to represent the same process. The Pentagon has already been identified as representing the conjecturing process. Since The Triangles was found to load most heavily on Factor I, the main characteristic of this situation seems to be that of conjecturing. The correlations between the scores on The Triangles, and the scores on verifying, on redefinition, on ability, and on achievement however do indicate that The Triangles as a composite problem having characteristics of set (the expectation toward area and perimeter that was held by the students) and substantiation (as in the verifying problems).

Analysis of the Process of Sensitivity

Do the scores on each sensitivity question relate to the other scores on the same question? Does the first sensitivity question relate to the second sensitivity question? To what extent is the sensitivity process identified by the two constructed problem situations? Table 16 presents the correlations between pairs of scores on the two sensitivity problems.

The correlations between pairs of scores on Sensitivity I, The Square are significantly different from zero. This is also the case for Sensitivity II. As for the conjecturing problems, correlations between fluency and novelty were lower than those relating either novelty and variety or fluency and variety. It may mean that the application of some system of categorization appears to introduce an influential factor into the measure or it may mean that the more fluent individual does not always produce

TABLE 16

INTERCORRELATIONS AMONG THE FLUENCY, VARIETY AND NOVELTY SCORES FOR THE TWO SENSITIVITY PROBLEMS

	Sen. I Fluency	Sen. I Variety	Sen. I Novelty	Sen. II Fluency	Sen. II Variety	Sen. II Novelty
Sen. I Fluency	1.00	-	-	-	-	-
Sen. I Variety	.66**	1.00	-	-	-	-
Sen. I Novelty	.39*	.73**	1.00	-	-	-
Sen. II Fluency	.11	.17	.13	1.00	-	-
Sen. II Variety	.02	.09	.14	.88**	1.00	-
Sen. II Novelty	-.02	.03	.10	.66**	.85**	1.00

* $p < 0.05$
** $p < 0.01$

the more novel responses. Again, as for conjecturing, it seems to be appropriate to use one score, probably variety, to describe each situation.

The two problems constructed to measure sensitivity do not correlate ($0.03 \leq r \leq 0.18$). Under both the six-factor image analysis (page 160) and the five-factor analysis (Appendix B), the two problems loaded on separate factors, relating more closely to other problems than to each other. Further description of each is dependent upon correlations to be reported later in this chapter.

Table 17 presents the correlations between pairs of scores on sensitivity, redefinition and verifying. As indicated earlier, finding ways to cut a square in half (Sensitivity I) at least as interpreted by the student response, appears to reflect a different process than does increasing the sides of a rectangle by ten percent. Some reason for this separation may result from an examination of the other correlations. Table 14 reveals that the fluency score for Sensitivity I, The Square, is unrelated to the conjecturing situations whereas the variety and novelty scores are related to The Pentagon, Conjecturing I. This discrepancy between fluency and novelty, or variety scores may be due to the various interpretations taken by the students toward the instructions. The instructions which required the students to cut the square in half and to illustrate the resultant halves were intentionally stated so as not to suggest the number of responses. As a result some of the students listed as many responses as possible, others listed as many different responses as possible. For example, some students gave the response

TABLE 17

CORRELATIONS BETWEEN PAIRS OF SCORES FOR THE SENSITIVITY, REDEFINITION, AND VERIFYING PROBLEMS

	Red. I	Red. IIA.	Red. IIB.	Ver. Ia.	Ver. Ib.	Ver. Ic.	Ver. Id.	Ver. IIA.	Ver. IIB.
Sen. I Fluency	-	-	-	-	-	-	-	-	-
Sen. I Variety	-	.43**	-	.35*	-	-	.34*	-	.54**
Sen. I Novelty	-	-	-	-	-	-	-	-	.40**
Sen. II Fluency	.42**	.37*	.32*	.38*	-	.39*	.40**	-	.53**
Sen. II Variety	-	-	-	.31*	-	-	.36*	-	.39*
Sen. II Novelty	-	-	-	.29*	-	-	.30*	-	.34*

* $p < 0.05$
** $p < 0.01$

⧄ meaning to include ⧅ and ⧄ ; others listed both. The former individual may have only realized one of the diagonal responses, and not considered it further. Another individual may have evaluated the response, ⧄ , realized that it was equivalent to ⧅ and thereby listed only the one response to represent the category of diagonal cuts. Thus the same response may have been made at two different levels of thought. These two levels are not distinguished by the scoring scheme for fluency. This evaluative work by the student, although also potentially influential in the conjecturing question, was not as obvious and thereby may have occurred less frequently in the responses to latter questions.

The correlations ($r = 0.32, 0.42$) between the novelty scores for The Pentagon and the variety and novelty scores for The Square are greater than any of the correlations between The Pentagon and Sensitivity II, Area-Increase (r is not significant). The first sensitivity question (The Square), in contrast to the second sensitivity problem, may have been interpreted as a divergent situation. Interpretation of the second sensitivity problem, Area Increase, as a convergent type problem, is supported by the significant correlations between the scores on this problem and the verifying and the redefinition problems ($r = 0.31, 0.53$).

Before the Area-Increase situation can be treated as a sensitivity question, there is a need to transcend the stage of verifying the idea that for each rectangle a ten percent increase in the length would result in a ten percent increase in area. This level of verification was the first level at which students operated. Those who searched the problem beyond that level had to overcome a set inherent in the wording of the question. "Two sides of a rectangle" was usually interpreted to mean two lengths or two widths. As a result the dimensions of the rectangle were increased, but in such a way as to maintain the rectangle. This interpretation is frequently made by texts, and many good students kept this justifiable interpretation. The breaking of this expectation is necessary before the sensitivity or the awareness aspect can be called into play. The initial stage seems to have been the stage at which most individuals operated. The existence of the three stages--verification of the given statement, redefinition of the meaning of the rectangle, sensitivity to the continuous relationship between side and area--in the problem may assist in interpreting the significant correlations present between Area-Increase, fluency, and redefinition (0.32-0.42) and between Area-Increase and Verifying (0.29-0.53).

The fluency score for Area-Increase correlates with redefinition while the variety and the novelty scores do not. For those individuals who broke the set and investigated the several possibilities resulting from an increase of two sides (and therefore received higher novelty and variety scores) the simple relationship between this question and redefinition was no longer found to exist.

The significant correlations between sensitivity II, fluency and redefinition would also follow from the observation that most of the responses to the former were manipulations with numbers in the area formula. Individuals verified the statement relating side increase with area increase by choosing specific numerical data for the sides of a rectangle in order to substantiate the percentage increase in area. The redefinition questions required the calculation of specific numerical data.

This approach of specific examples used to verify the Area-Increase relationship was also used by many students to justify the area relationships stated in The Polygons, Verifying I. This was specifically true for parts a and d which dealt with the side-area relationship in the square and in the general polygon, respectively. The latter was often justified in terms of the specific case of the square. The testing of specific examples in the Area-Increase situation may also explain some of the high correlation between the variety score of the sensitivity situation and the specific part of the second verifying situation, Parallel Lines, which involved using specific angle measures in their relationship to parallel lines and transversals.

There appears to be a relationship between Redefinition II, Angles, and Sensitivity I, The Square, Variety. It has been hypothesized that some type of redefinition is necessary to break the set that all halves are not formed by straight lines. If the high correlation between the variety score and the redefinition situation ($r = 0.43$) reflects a redefinitive aspect in both situations, it would be expected that the novelty score would also correlate with the redefinition question or that The Square would correlate with The Board and Hole problem, or with Redefinition II, Areas. This was not found to be the case. It may be that the further categorization and statistical drawing of boundaries for originality necessary for the novelty score has introduced an aspect that overpowers this relationship. However, there is no reason to expect that the type of "breaking of set" that is required when solving a familiar problem on area or angle size, with a knowledge of the type of solution required, is the same as the "breaking set" required to realize that it is possible to cut a square in half by other than straight line cuts, much less the same as the ability to finalize novel means for cutting this square in half.

Table 18 presents the correlations between scores on the sensitivity problem situations and the scores for achievement, for ability (verbal and non-verbal), and for the Torrance tests of creativity. Neither of the two sensitivity problem-situations correlated with scores on the Torrance tests. Again as with the conjecturing situations, the sensitivity situations measured processes that did not relate to those required to generate responses to fanciful situations. (See discussion pages 158 and 161)

The scores on Sensitivity I, The Square, were found not to correlate with intelligence as measured by the Lorge-Thorndike tests or with ability as measured by SCAT, except for the 0.38 correlation of Sensitivity I, variety with the SCAT nonverbal scores. A variety score also correlated with the achievement test ($r = 0.38$). The categorization used in scoring the first sensitivity question for variety may have resulted in the significant correlation between this score and achievement, with the nonverbal ability score. This significant correlation was not found to exist for the fluency score, that is, before the classification was made. The correlation between

TABLE 18

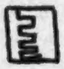
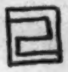
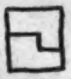
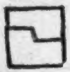
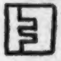
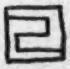
CORRELATIONS BETWEEN PAIRS OF SCORES FOR THE SENSITIVITY PROBLEMS AND THE STANDARD MEASURES OF ABILITY, ACHIEVEMENT AND CREATIVITY

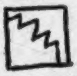
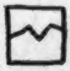
	SCAT V	SCAT Q	MATH ACH.	NVCR. F.	NVCR. FLEX.	NVCR. ORIG.	NVCR. ELAB.	VER. F.	VER. FLEX.	VER. ORIG.	LORGE THORNDIKE V.	LORGE THORNDIKE N.V.
Sen. IF	-	-	-	-	-	-	-	-	-	-	n	n
Sen. IV	-	.38*	.38*	-	-	-	-	-	-	-	-	-
Sen. IN	-	-	-	n	n	-	-	-	-	-	-	n
Sen. IIF	.43**	.59**	.53**	-.46**	-.44*	n	n	n	n	n	.50	.36
Sen. IIV	.44*	.39*	.46**	-.45*	-.44*	n	-	n	n	n	.39	-
Sen. IIN	.34*	-	.41*	n	[illegible]	-	-	n	-	-	-	n

* $p < 0.05$
** $p < 0.01$
n indicates a negative but insignificant correlation

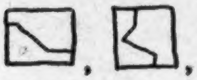
Legend: F--Fluency
V--Variety
N--Novelty
NVCR.--nonverbal creativity score
Ver.--verbal creativity score

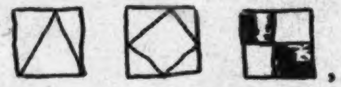
achievement and novelty was also not significant. The final grouping or categorization of responses used to establish the novelty score may have resulted in some bias affecting this variation in the correlations. More specifically, the following groupings may be questioned:

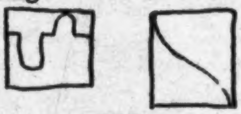
i)  was grouped with  and  . As a result this group contained a large number of responses, causing these responses to be scored low on novelty. (Fourteen individuals made the response  , seven made one of the two or both of the responses  ,  .

ii)  was made by fourteen individuals.  was placed in the same category, but was only made by three individuals. These were placed in separate categories for the variety scoring, as were the responses discussed under i).

iii) Other categories, which were made from responses separately grouped for the variety measure, are the following:







I II III

These additional groupings in the novelty measure were necessary to obtain a range of novelty scores from 1-4. Without the extra categorization many scores of 4 would be given and many scores of 0 would be given.

The second sensitivity situation correlated significantly with measures of ability, (0.43-0.59), and with measures of achievement (0.41-0.53). The children apparently did not see the second sensitivity question as an open-ended situation. Even when prompted, no one extended the "rectangle" to its limit points of a parallelogram which is almost flat, or examined the possible shapes of a quadrilateral which has the specified sides. In fact, no one saw that the answers lay along a continuum. As a result the fluency score was not really a fluency score as measured by Torrance's situational or straight line tests in which students know that there are an infinite number of answers.

The response to this question implied that no one recognized the second sensitivity question in its "sensitive" character. This may have been due to two factors. First, the level of mathematical competency and computational facility held by grade nine students was not sufficiently high. Students were unable to proceed to the generalizations quickly enough to realize the open-ended nature of the question. Second, there was a necessity for the students to be attentive to a subtlety in word meaning (two sides). Instead the attention to the straight application of facts when responding to the second sensitivity situation was emphasized and this may explain the high correlations with standard achievement and ability test scores. Because of the complications in the second sensitivity situation, Area Increase, this problem may not be an adequate measure of sensitivity for students of this level of mathematical sophistication.

Summary.

The different measures of fluency, variety and novelty on each question correlated, and therefore only one of these measures may be used to represent the performance on sensitivity. The same arguments used for favoring the variety score on the conjecturing process may also be applied here. (See page 178 of this chapter)

The two sensitivity problems did not correlate, re-emphasizing the loadings on separate factors found under image analysis. The first sensitivity question, The Square, which loaded on Factor VI, seems to be the better representation of sensitivity. It correlated to the conjecturing question and to the redefinition questions, reflecting the redefinitive and the divergent characteristics of sensitivity outlined in the guidelines. (Chapter III, pages 59-61) The second sensitivity question, Area-Increase, however seems to be a composite of various traits shared by the redefinition, the verification, and the ability tests. The students treated this question as they had the convergent problems. The realization of other possibilities and therefore the production of these other possibilities for the question did not materialize because of the difficulty of the question. As a result, Factor VI will be considered the sensitivity factor and The Square will be assumed to reflect this process.

Analysis of the Process of Redefinition

Table 19 presents the correlations among the redefinition scores and the verifying scores. This table together with Table 14 (page 180) and Table 16 (page 189) present the correlations between scores on redefinition and on the other processes.

TABLE 19

CORRELATIONS BETWEEN PAIRS OF SCORES ON THE REDEFINITION PROBLEMS AND THE VERIFYING PROBLEMS

	Red. I	Red. IIA.	Red. IIB.	Ver. Ia.	Ver. Ib.	Ver. Ic.	Ver. Id.	Ver. IIA.	Ver. IIB.
Red. I	1.00	.60**	-	.40**	.42**	.50**	.38*	.60**	.38*
Red. IIA.	.60	1.00	.44**	.46**	.39	.41**	.53**	-	.49**
Red. IIB.	.24	.44**	1.00	-	-	-	-	-	-

* $p < 0.05$
** $p < 0.01$

The three redefinition scores were found to correlate significantly with each other. This fact was reflected in the image analysis results which showed Redefinition I, IIA, IIB loading primarily on Factor III. The interrelationships among the redefinition questions may be summarized as follows:

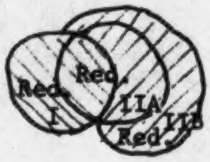
Redefinition I correlated with Redefinition IIA. ($r = 0.60$)

Redefinition IIA correlated with Redefinition IIB. ($r = 0.44$)

Redefinition I did not correlate significantly with Redefinition IIB. ($r = 0.24$)

It seems that the processes necessary to solve Redefinition IIA are described by those needed to solve Redefinition I and those needed to solve Redefinition IIB. However very little is common to all three.

A Venn diagram might show the relationship between The Board and Hole, Redefinition I, Redefinition IIA, The Angles, and Redefinition IIB, The Areas.



Redefinition IIA and Redefinition IIB were structured in a similar way and were administered as one situation. The first deals with angle measures, the second with areas. In both the attempt is to reinforce the finding of a solution to a problem similar to that used in the classroom, then to place new situations before the respondent. These new situations call upon the student to alter his method in order to find a more efficient one for solution.

The relationship between Redefinition I, The Board and Hole, and Redefinition IIA, Angles is a strange one in light of the non-significant correlation between The Board and Hole and Redefinition IIB, The Areas. Initially it was supposed that rearrangement of shapes to find areas would correlate with the ability to cut a board in half and to rearrange the halves to form a new area. This was not observed to happen.

Redefinition I, The Board and the Hole, was found to not correlate with The Square, although both situations deal with cutting a rectangle in half. The lack of correlation may be because individuals who responded well to the breaking of a set in an open situation may be different from those individuals who respond to the breaking of a set in a highly directive situation. Only fifteen of the forty-two individuals were able to solve The Board and Hole problem. All fifteen of these individuals were able to produce three or more types of responses to The Square. However, the five individuals who scored zero on The Board and the Hole produced five or six different responses to The Square. These individuals were often flustered by the specific requirements of the former question, but could whimsically experiment with The Square.

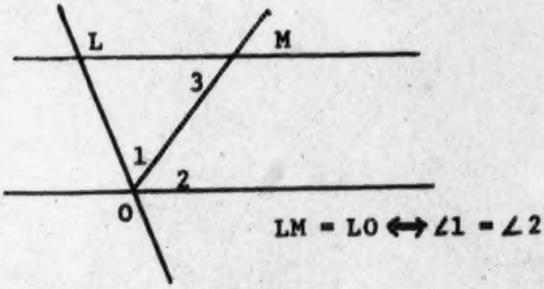
During the experiment it was realized that, because of the similarity, The Board and the Hole, if done first, could sensitize a student to other possibilities in The Square, or vise-versa. The Square was administered after The Board and the Hole to all students except for the few whose schedule had been interrupted by absenses and therefore the effect should have been the same for all. The oral interview, however, revealed that some of the students realized the relationship between the two situations while others did not even remember The Board and Hole when they were confronted by The Square. A student with ability to relate one problem to another may have scored higher on the problem than he would have had he worked the problem in isolation. This relative effect of previous experience may have distorted some of the correlations.

Redefinition I, The Board and the Hole, and Redefinition IIA, the Angles, were found to correlate significantly with the verifying situations, the heavy correlations resulting in a loading of 0.32 on Factor I as well as the 0.70, 0.61 loadings on Factor III, the redefinition factor. This high correlation was emphasized by the results of the five-factor analysis where the redefinition and the verifying questions all loaded on one factor. This separation of these questions from those of conjecturing and sensitivity would suggest that the students saw at least two types of questions and may therefore justify the separation of the processes into convergent and divergent branches.

The strong relationship between The Board and Hole problem and the general version of the parallel lines problem may be due to a similarity in the complexity of thought required for solution. Both questions required the student to work with two ideas at one time. To solve the general parallel line problem the student had to realize i) that $\triangle LMO$ was isosceles and therefore $\angle 1 = \angle 3$.

ii) that the line MO was not only a side of a triangle but that it was also a transversal, thus $\angle 3 = \angle 2$.

iii) that if $\angle 1 = \angle 3$, $\angle 3 = \angle 2$, then $\angle 1 = \angle 2$.



The students solving The Board and Hole problem were required to work with the following information: i) the board was to be cut in half with only one cut; ii) the specific numerical ratios of 15:10, and 3:2 for the dimensions of the rectangle--a fact which seemed to inhibit those having difficulty, yet at the same time stimulating in others the means for a quick solution.

When specific angles were stipulated in the parallel lines question, the students no longer had to keep one idea in mind while working with another. They could calculate angle measure one step at a time, then notice that the required angles were equal. The correlation between this part of the problem and The Board and Hole problem was consequently less.

Table 20 presents the correlations of the scores on the measures of redefinition with the scores on the standard measures of creativity, ability and intelligence, and achievement.

The redefinition tests were found to not correlate with measures of creativity, the Torrance tests, except for the one correlation ($r = 0.31$) between Redefinition II^{B}, Angles, and nonverbal originality. This correlation may be attributable to the difficult item (item 6) on Redefinition II, Angles. This item which was solved by only two individuals required the student to find the area of a arrow shaped polygon by rearranging it into a parallelogram. The two individuals tended to perform better on the Torrance tests than did the rest of the sample. Other than that one correlation, it seems that the Torrance tests and the redefinition tests measure different processes or abilities.

Redefinition I, The Board and the Hole was found to relate to the standard measures of ability ($r = 0.39$, 0.44) and of achievement ($r = 0.42$). The individual who performs well in accepted tasks in the classroom was also able to rearrange a 3 x 10 rectangle into a 2 x 15 rectangle by cutting the given one in half. Similarly the questions on angle measure and on findings areas correlated highly with measures of ability. ($r = 0.56$, 0.59) The

TABLE 20

CORRELATIONS BETWEEN PAIRS OF SCORES ON THE REDEFINITION, VERIFYING AND THE STANDARD MEASURES

	SCAT V	SCAT Q	MATH. ACH.	NVCR. F.	NVCR. FLEX.	NVCR. ORIG.	NVCR. ELAB.	VER. F.	VER. FLEX.	VER. ORIG.	LORGE THORNDIKE V.	LORGE THORNDIKE N.V.
Red. I	.39*	.44**	.42**	n	n	n	-	n	n	n	.39*	.49**
Red. IIA	.56**	.59**	.53**	n	n	-	n	n	n	n	.57**	-
Red. IIB	.38*	.46**	-	-	-	.31*	-	n	n	n	.48**	-
Ver. Ia	-	.41**	.39*	n	n	n	n	n	-	n	-	n
Ver. Ib	.30*	-	.30*	n	n	n	-	n	-	-	-	-
Ver. Ic	-	.44**	.50**	-.38	-.36	n	n	-	-	-	-	-
Ver. Id	.33*	.50**	.53**	-.33	-.40	n	n	-	-	-	-	-
Ver. IIA	.31*	.37*	.44**	n	n	n	-	-	-	-	-	.31
Ver. IIB	.51**	.50**	.51**	n	n	-	-	-	-	-	.36	.39

* $p < 0.05$
** $p < 0.01$
n indicates a negative but insignificant correlation

Legend: F--Fluency
V--Variety
N--Novelty
NVCR.--nonverbal creativity score
Ver.-verbal creativity score

angle question also correlated with achievement. (r = 0.53) The area question did not correlate with achievement, possibly because the specific approach and not the answer was awarded the high marks. Although this type of scoring was used for Redefinition IIA, Angles, the separation of clearly appropriate answers was not made to the same degree. It may be that the approach awarded the higher score was more appropriate than the one awarded the lower score.

Summary.

Two of the three redefinition questions, The Board and the Hole, and Angles were found to correlate significantly different from zero and to characterize Factor III most strongly. All three problems related sufficiently to load above the 0.30 level on this factor, which seems to represent the process of redefinition.

Redefinition IIB, Areas seems to reflect some other processes as well. These processes do not seem to be described by the six-factor solution, since the total communality shown for this test on the six factors was 0.30, totally due to the loading on Factor III.

Redefinition I, The Board and the Hole and Redefinition IIA, Angles, correlated with the verifying situations, and to parts of the second conjecturing questions. The close relationship between verifying and these two redefinition problems is reflected by the high loading of their scores on one factor when a five-factor solution for the image analysis was required. This relationship may be due to the strong common trait of convergence.

Factor III, characterized by Redefinition I and Redefinition IIA seem to represent the redefinition process. This may be summarized as the ability of manipulating data already in the question so as to achieve a most appropriate solution. Redefinition IIB seems to reflect this characteristic to a lesser degree.

The scores on Redefinition I and Redefinition IIA were found to correlate with the ability and the achievement scores. It may be assumed that the questions of redefinition measure somewhat the same processes as do the ability tests, or that the individual who is capable of redefining ideas and rearranging them is the same individual who does well in the tasks assigned in the classroom. The student who is able to work the problems, to rearrange the numbers in the usual application questions is possibly the more flexible and the more alert for an easier solution to a given problem.

Analysis of the Process of Verification

The first verifying situation was found to correlate very highly within itself. (Table 21) This suggests that perceiving that the area of the square increases as the square of the side increases relates closely with the ability to perceive the same relationship in a triangle, in a pentagon, or in a general polygon. This does not mean that those individuals who were able to present some rationale for the relationship in a square were able to do so for other polygons, but an individual who was able to perceive the

TABLE 21

INTERCORRELATIONS AMONG THE SCORES ON THE VERIFYING PROBLEM SITUATIONS

	Ver. Ia	Ver. Ib	Ver. Ic	Ver. Id	Ver. IIA	Ver. IIB
Ver. Ia	1.00	-	-	-	-	-
Ver. Ib	.58**	1.00	-	-	-	-
Ver. Ic	.71**	.65**	1.00	-	-	-
Ver. Id	.71**	.48**	.70**	1.00	-	-
Ver. IIA	.33*	.42**	.47**	.40**	1.00	-
Ver. IIB	.35*	-	.38*	.35*	-	1.00

* $p < 0.05$

** $p < 0.01$

relationship in a general sense was able to do the specific parts in an abstract way.

The high correlation between part (a), the area of a square, and part (d), the area of a general polygon, may be due to several relationships seen in the list of scores awarded to the individuals. First of all, some individuals proceeded to prove the general case using a square. Secondly, lower scores awarded to (a) corresponded to zero scores on (d). Ranks of one and two (the highest rank awarded was 6) were given to those individuals who were able to give only one or two specific examples to substantiate the statement. These individuals then supposed the statement to be proven for all cases; and orally were not able to generalize the abstract argument. These individuals were not able to cope with the abstract argument necessary in part (d) for the area of a general polygon. Many of these did not even realize that part (d) was a generalization of parts (a), (b), or (c). For some, part of the difficulty was an incomplete understanding of the term polygon.

Those individuals who scored anything but zero on part (d) tended to obtain a higher rank on part (a).

Only two of the forty-two individuals scored higher on part (d) than on part (a). Both individuals seemed more concerned with solving the abstract statement, being aware that the others were specific parts, and produced in incomplete solution which promised, in the scorer's opinion, an intuitive general proof. One individual chose to work with the circle for part (d) without spending any time on the first three parts, but because of the difficulties which arose from dealing with large numbers and with π, he was unable to complete any formalized statement. He received a high score for this incomplete proof, and, not having spent much time on the other parts, scored low on the first three parts--2 on (a), 0 on (b) and (c). The second individual spent more time on each specific part but also realized that the extreme generalization of a polygon would be a circle. He also did not complete the proof for this idea within the given time limit. This latter individual appeared more adept in computational skill and was later able to complete this proof.

The discrepancy observed in these two cases may be the fault of the way the problem was scored. A more complete scoring scheme to deal with problems like these, which may occur more frequently in a larger sample, might possibly assign the individual the score he received on the last part, if this score was the greatest, to all other previous parts of the problem as well. In other words Part (a), Part (b), Part (c) $\subseteq$ Part (d).

It is possible, however, that the incomplete solution, especially in the first individual's case, might not rate such a high rank. It may be the experimenter's interpretation that awarded this response its promise. The individual may not have been able to completely generalize from the circle to the general polygon. He did realize that the circle was an extreme case, however may not have been able to explain the connection between the radius of a circle and the side of a polygon.

Of the fourteen individuals who scored 0 or 1 on parts(a) of The Polygons problem only one received higher than this on any of the other three parts. This individual was able to work with the triangle, possibly because of carry-over from the discussions on the second conjecturing question, with which he greatly identified. He spent little time on the square, and was not able to generalize the concept of area and altitude to all polygons and received 0 on all other parts of the first verifying problem.

The correlation between part (a) and part (b) ($r = 0.58$) of The Polygons was moderate and significant but not as high as that between (a) and (c) ($r = 0.71$) or that between (a) and (d) ($r = 0.65$). This reduced correlation may be due to the following reason: most students who attempted part (a) also attempted part (b) of the problem because it dealt with a familiar polygon. However because the correct solution required the intermediate understanding of the altitude as related to but distinguished from the side of a polygon, many students who had achieved a solution to part (a) at an intuitive level did not achieve an equivalently correct solution to part (b).

This distinction between side and altitude is not explicitly distinguished in the square problem and many students solving part (a) did not deal with these concepts and thus were unable to discuss i) the necessity of preserving similarity and ii) the proportionality between side and altitude under this provision. In conclusion, the smaller correlations between part (b) and the other parts of the question could be at least partially due to many individuals attempting a familiar situation which they did not fully grasp, and for which they used assumptions that were not questioned

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or perhaps not even realized.

A high correlation was found between parts (a) and (c) ($r = 0.71$). Only those individuals who did well abstractly with the square attempted and dealt with the pentagon; those who could only cope with part (a) at the specific level seldom even attempted part (c). Thus the scores awarded the pentagon were either zero or high.

The correlation between parts (a) and (d) ($r = 0.65$) can be explained in the same way as that between (a) and (c). Similarly the separation of scores into two groups--0 or high--would at least partially account for the 0.70 correlation between (d) and (c).

The following discussion will refer to the correlations among scores on the verifying measures and the scores on the other constructed problems found in Table 14 (page 180), Table 16 (page 189), and Table 20 (page 205).

High correlations ($0.31 \leq r \leq 0.48$) were found between Conjecturing II (variety) and the verifying problems. This may have been due to the students conceptualization of The Triangles as a verifying type of situation. The students began to examine the relationship between the area and the perimeter of the triangle and became engrossed with finding out why area varied even if the perimeter remained constant. The students who made responses which were more complex than just observable facts and which tied several concepts together, received higher variety scores. These students usually did well in the first verifying situation which dealt with side-area concepts.

Most of the students, however, did not extend themselves beyond the area-perimeter sphere for The Triangles, and thus their responses were not statistically novel. The reduced relationship between the scores on the verifying situation, The Polygons, and the novelty score for Conjecturing II, The Triangles reflect this fact.

The common content element present in Verifying Ib and in Conjecturing II may further account for the correlation ($r = 0.48$) between the two questions. Both dealt with the area-perimeter concept in triangles.

A high correlation also exists between the variety score on Conjecturing II, The Triangles, and Verifying IIA, Parallel Lines, Abstract Argument ($r = .43$). The high relationship does not extend to the more specific part of Parallel Lines. It may be that because of the specific numbers included in the second part of Parallel Lines, the student was able to blunder and to work through the angle measures without having to carefully examine and manipulate several variables at one time. This ability to retain and to manipulate several variables in order to pose a possible relationship is present in both Conjecturing II and in Verifying IIA. In fact, the distinctive feature of Verifying IIA, Abstract Argument, may be the presence of the two independent ideas, first that relationship between parallel lines, transversals, and the angles thus formed, and second, that of the isosceles triangles and its relationships. A successful response was possible only when both ideas were combined and used by the student. In the conjecturing question, the variables of area and perimeter depended on altitude, shape and angle size. Granted, in the conjecturing situation, the direction in which manipulation occurred was open to students, whereas in the verifying problem a distinct required answer directed student efforts. However, since the level at which most students approached these problems was very intuitive, this distinction may well be hypothetical rather than actual. Such a distinction would only make itself apparent to the student who had some idea of where the questions led, who had advanced beyond the "floundering" level.

Verifying IIA, Abstract Argument, loads on two factors, that shared by the other verifying situation ($r = 0.36$) and that shared by the redefinition questions ($r = 0.5$). It is strongly associated with the redefinition questions, and this may be the result of the manipulative ability necessary to work with several variables at once.

The scores from Verifying IIB the specific part of Parallel Lines, correlate closely with the redefinition scores, and load on Factor III ($r = 0.52$) which represents the redefinition process. This problem differs from Parallel Lines, part A, by the inclusion of specific numerical values for angle measures. This use of the numbers resulted in a closer correlation of this problem with redefinition. The quality of manipulation of specific numerical data seems to meet the requirement of definition; it seems that this question was misnamed and should have been classified as redefinition.

Table 20 presents the correlations between pairs of scores on the verifying problem-situations and the standard measures of ability, achievement and creativity. The significant negative correlations between the Torrance Tests and the verifying situations indicate the opposing qualities of divergence and convergence in the two situations. All students were able to respond to the situations presented by the Torrance tests whereas many students were awarded a score of zero on the verifying situations. That is, on the verifying problems, no distinction was made between the student who didn't respond and the one who responded with an inappropriate statement.

The verifying situations were found to correlate significantly with the standard measures of ability and achievement. This may mean that the constructed problems measured to some extent a process already present in established measures of student success in the classroom. This is not surprising as the process of verification is emphasized in the classroom. The correlation between the first verifying situation, The Polygons, and the ability tests were not large enough to influence the image analysis, and SCAT did not load on Factor I.

Summary.

The scores on the parts of Verifying I correlate with each other and with the scores on Verifying II. Verifying IIB does not correlate with Verifying IIA and does not correlate as highly with Verifying I as does Verifying IIA. As a result it seems that Verifying I, The Polygons, and Verifying IIA, the abstract part of Parallel Lines are a consistent representation of the same process.

The scores from these questions were found to load on Factor I, which seems to represent the verifying process. Verifying IIB, the specific part of Parallel Lines, seems to be a redefinition question. This follows for two reasons. First the question seems to meet the qualities possessed by the other redefinition questions. The problem presents specific numerical data to the student for reorganization into a specific and correct solution. This trait also meets the guidelines for redefinition more appropriately than it meets the guidelines for verifying. Second, the correlations between redefinition and Verifying IIB are higher than the correlations between the other verifying questions and Verifying IIB.

The verifying questions were found to correlate with the redefinition question. Both types of problems measure the convergent processes, and both questions seem to relate to the type of thinking presently emphasized in the classroom. This latter fact is supported by the high correlations found between the scores on the verifying questions and the scores on the ability and achievement tests.

Summary of Results

Table 22 presents a summary of the relationships found between the twenty-three scores used in the image analysis. (The Torrance tests were not included in the Table since no relationship occurred between the scores on these tests and the scores on the

TABLE 22

A SUMMARY OF THE RELATIONSHIPS PRESENT AMONG THE PROBLEMS

<table>
<tr><th></th><th></th><th>HYP. I</th><th>HYP.II</th><th>SEN. I</th><th>SEN. II</th><th>RED. I</th><th>RED. II A</th><th>RED. II B</th><th>VER. I</th><th>VER. IIA</th><th>VER. IIB</th><th>SCAT V</th><th>SCAT NV</th><th>MATH ACHIEV.</th></tr>
<tr><td>HYP I</td><td>F</td><td>*</td><td>*</td><td></td><td></td><td>NIL</td><td>NIL</td><td>NIL</td><td>NIL</td><td>NIL</td><td>NIL</td><td>NIL</td><td>NIL</td><td>NIL</td></tr>
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<tr><td>HYP II</td><td>F</td><td>*</td><td>*</td><td></td><td>NIL</td><td>*</td><td>*</td><td></td><td>*</td><td>NIL</td><td></td><td>*</td><td>*</td><td>*</td></tr>
<tr><td></td><td>V</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>*</td><td>NIL</td><td>*</td><td>*</td><td></td></tr>
<tr><td></td><td>N</td><td></td><td></td><td>*</td><td></td><td></td><td></td><td>*</td><td></td><td>NIL</td><td></td><td>*</td><td>*</td><td>*</td></tr>
<tr><td>SEN I</td><td>F</td><td></td><td></td><td>*</td><td>NIL</td><td>NIL</td><td></td><td>NIL</td><td>NIL</td><td>NIL</td><td>*</td><td>NIL</td><td>NIL</td><td></td></tr>
<tr><td></td><td>V</td><td></td><td></td><td></td><td></td><td></td><td>*</td><td></td><td></td><td></td><td></td><td></td><td></td><td>*</td></tr>
<tr><td></td><td>N</td><td>*</td><td>*</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>SEN II</td><td>F</td><td></td><td>NIL</td><td>NIL</td><td>*</td><td>*</td><td>*</td><td>*</td><td>a b c d
* | | | *</td><td>NIL</td><td>*</td><td>*</td><td>*</td><td>*</td></tr>
<tr><td></td><td>V</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>* | | | *</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>N</td><td>*</td><td></td><td></td><td></td><td></td><td></td><td></td><td>* | | | *</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>RED I</td><td></td><td>NIL</td><td>*</td><td>NIL</td><td>* |-| |</td><td>*</td><td>*</td><td>NIL</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td></tr>
<tr><td>RED II</td><td>A</td><td>NIL</td><td>* | * | *</td><td>| | |</td><td>* |-| |</td><td>*</td><td>*</td><td>*</td><td>*</td><td>NIL</td><td>*</td><td>*</td><td>*</td><td>*</td></tr>
<tr><td></td><td>B</td><td>NIL</td><td>* |-|</td><td>NIL</td><td>* |-| |</td><td>NIL</td><td>*</td><td>*</td><td>NIL</td><td>NIL</td><td>NIL</td><td>*</td><td>*</td><td>NIL</td></tr>
<tr><td>VER I</td><td></td><td>NIL</td><td>*</td><td>NIL</td><td>-</td><td>*</td><td>*</td><td>NIL</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td><td>*</td></tr>
<tr><td>VER IIA</td><td></td><td>NIL</td><td>| * | |</td><td>NIL</td><td>NIL</td><td>*</td><td>NIL</td><td>NIL</td><td>*</td><td>*</td><td>NIL</td><td>*</td><td>*</td><td>*</td></tr>
<tr><td>VER IIB</td><td></td><td>NIL</td><td>NIL</td><td>*</td><td>*</td><td>*</td><td>*</td><td>NIL</td><td>*</td><td>NIL</td><td>*</td><td>*</td><td>*</td><td>*</td></tr>
</table>

other tests.) The research questions posed at the beginning of the chapter were the following:

i) Do the scores from the constructed tests cluster into four distinct groups which may be interpreted in terms of the model? Do the different scores for each question correlate?

ii) Do the scores for the problems representing the same process correlate?

iii) Do the scores for the problems on one process correlate with the scores for problems on another process?

iv) Do the scores for the constructed problems correlate with the scores for the standard measures of ability, achievement and creativity?

The last three questions are answered first. The answer to the first question includes some conclusions about the results.

i) The different scores for each question correlated at a high level. This would indicate that one measure would be appropriate for each question. Specifically the added categorization of responses to achieve the variety and novelty scores for the divergent problems did not basically change the process reflected by the question. The variety score is considered to incorporate best the requirements of appropriateness and objectivity, and thus to be the best representation for this study of the performance on the divergent-process problems.

The scores on one problem representing a process did not always correlate with another problem for the same process. This lack of correlation was most notable for sensitivity. Verifying IIB also did not correlate with other measures of verifying but this is considered to be due to an error in classifying this problem. It seems that Verifying IIB is actually redefinition.

ii) The scores for the constructed problems on one process correlated to some extent with scores from another process. Conjecturing I, The Pentagon, was an exception showing little correlation with the other constructed problems. Similarly, The Square, Sensitivity I, seemed to remain distinct from most other problems. The Triangles and Area-Increase on the other hand seemed to be conglomerate problems. The problems for verification correlated with the problems for redefinition. These associations helped to identify the characteristics in each problem.

The separation of these scores on orthogonal factors would, however, support the contention that the hypothetical processes of conjecturing, sensitivity, redefinition and verifying may be independent but that problems calling for only one process, and not another, cannot be constructed at a non-superficial level.

iii) To a large extent, the constructed problem-situations correlated with ability and achievement. Conjecturing I seems to measure a process not reflected by the standard measures. Sensitivity I also seems to remain distinct from ability and achievement. The other problems related to some extent with the standard measures, redefinition and verifying most notably. The extent of the correlation depended upon the specific problem involved. These correlations helped to identify the characteristics of each problem.

iv) The scores from the constructed tests clustered in five groups, all distinct from the scores on the Torrance tests which clustered on the sixth factor. Four of these five clusters were interpreted as conjecturing, verifying, redefinition and sensitivity. The other factor was characterized by Sensitivity II, but this problem seemed to be a conglomerate of abilities and the factor could not be definitely identified. The following points identify the factors from the image analysis, and try to describe the problems in terms of these factors.

(a) The conjecturing process may be considered to be represented by Factor II. The pentagon problem largely characterized this factor. The interpretation of this factor in terms of the conjecturing process helps to describe the few correlations between performance on The Pentagon and on the other tests used in the study.

(b) The verifying process may be presented by Factor I. This factor was largely described by the Polygons, Verifying I.

(c) Conjecturing II, The Triangles, is a problem that can be described in terms of both verifying and conjecturing. Its loading on both Factor I and on Factor II is consistent with the experimenter's observations of student responses as well as with the interpretation of Factor I as verifying and Factor II as conjecturing.

(d) The process of redefinition appears to be approximated by Factor III. This factor was characterized by the tests of ability, the tests of redefinition, and by the second verifying situation. Factor III may thus be described as representing a convergent process, one in which numerical data is used to obtain a specific answer to a specific situation. The questions describing this factor require some reinterpretation of data and the discarding of seemingly obvious data or interpretations of this data in order to achieve a more efficient or more appropriate response.

e) The two sensitivity questions described different processes and loaded on two distinct factors. The Square (Sensitivity I) related to the conjecturing questions and to one of the redefinition questions. In this sense, it may meet the specifications for sensitivity, being divergent in character and requiring the breaking of a "set" in this case the set of "one-half". Sensitivity II, Area-Increase, was found to relate more closely to the problems reflecting convergent processes. This is thought to be due to the several levels present in the problem, and the fact that the level of solution at which sensitivity would occur was too difficult for the grade nine students in the sample.

CHAPTER VII

THE ORAL INTERVIEW

This chapter, considered to be the essence of the study, describes some of influences and some of the procedures used by the grade nine students in this sample in solving complex mathematical problems. This is knowledge that may be invaluable to any teacher of mathematics. He is confronted every day with a great number of children who are solving problems and many of them take devious routes to their solutions. It is important that a teacher understand some of the techniques, some of the conceptions (or misconceptions) and some of the interpretations that are present in the minds of the children if he is to communicate or to modify their mental behavior.

The chapter is divided into three sections, one dealing with conjecturing, another dealing with redefinition, and the third, with sensitivity. The content has been chosen so as to give the reader the greatest possible insight into the great mass of information which confronted the researcher. The greatest concentration is on the conjecturing process, a decision made for the following reasons:

i) Although the plan is to examine the tapes for all eight questions in equal detail, time considerations necessitated pausing at this stage.

ii) A close look at the conjecturing process appeared to offer the greatest amount of information about the divergent processes. The researcher wished to know whether grade nine students were capable of asking fruitful questions, and also the means by which they arrived at these questions.

CONJECTURING

This part of Chapter VII is divided into three sections. The first section presents some of the student responses. An effort was made to classify these responses into the framework established by Heinke in order to ascertain whether children of this age had developed intuitive ways of constructing conjectures that were similar to those used by mathematicians.

The second section presents taped responses to oral questioning. This record of responses revolves around individuals' approaches toward the problem-situations. An attempt is made to include i) a discussion of the categories of approach used by the students; ii) an examination of the flow and the sequencing of ideas which occurred in the thinking processes of these children; and iii) an indication of the extent to which children are aware of their conjecturing procedures.

The third section presents a look at the potential of the problem-situations in a classroom situation. It also gives the reader an idea of some of the mathematics inherent in the two problem-situations, thereby providing him with some basis for comparison for the student responses discussed in the first section of the chapter.

Means of Forming Conjectures According to Variation of Hypotheses by Heinke

The conjecturing problem attempts to separate and to reflect the individual's ability to formulate or to pose questions about information. Since the type and quality of conjecture formulated determine the information that will be advanced by investigating these questions, a most important task that must be performed by the creative mathematician is the expression of his explorations of a set of data into a form which directs further investigation, a statement whose truth value can then be determined and tested.

Heinke has outlined a detailed description of how one statement can be developed from an already established statement. This process, which he calls variation, would seem to be an elaboration of Freedman's suggestion that one conjecture often leads to many other productive ones; that mathematicians often proceed by varying, limiting, and extending hypotheses, examining the resultant effect on conclusions; or by varying conclusions in these same respective ways, thereby investigating the corresponding effect on the conditions.

The conjectures made by the students were categorized according to Heinke's classification. Table 23 presents the number of each type of conjectures made. Although this is just an approximate count, it does indicate that children can make conjectures and that they make them in ways that are considered valid by mathematicians.

TABLE 23

TYPES OF CONJECTURES MADE BY STUDENTS IN-SAMPLE

	SUBSTITUTION OF INFORMATION						OTHERS				
	DELET. OF DATA	ADD. OF DATA	VAR. OF SPEC.	GEN.	INDUC.	CHANGE REL.	CON-VERSE	INVERSE	CONTRA-POS.	DUAL PRIN.	VAR. OF DP.
					the PENTAGON						
VAR. of cond.	2	19	30	1	-	1	-	6	-	-	1
VAR. of conc.	1	5	11	1	-	-	-	-	-	-	-
OBSERVATIONS TOTALLED TO 57			TOTAL RESPONSES 125			The total number of responses was about 145, however some were irrelevant, some the author was unable to classify.					
					the TRIANGLES						
VAR. of cond.	1	18	14	1	1	10	4	9	-	2	2
VAR. of conc.	1	2	4	1	-	2	-	-	-	-	-
OBSERVATIONS TOTALLED TO 83			TOTAL RESPONSES 155			The total number of responses was about 130, however some of the responses were recorded twice since they seemed to fit more than one category.					

Please note that this tally was only an approximate count of how the conjectures were made, and therefore there is discrepancy in number.

Many of the responses (57/125, 83/130) were not true conjectures, but were classified as observational statements. Any individual who made only these statements scored zero on originality. The very obvious observations, such as "A triangle has three sides" were not considered as appropriate answers for any score.

Heinke sees a statement as do most mathematicians, as consisting of data, conclusion, and the relation between the two. Thus conjectures may be generated by merely manipulating verbally these three aspects of the statement. Some grade nine students were observed to operate this way. E.,who was a high achiever in language arts, did not construct original conceptual statements, but used variations of given conjectures by changing the words in the original statement. His comparison of conjecturing was to an analysis of poetry, where one examines the structure of the words and their arrangement within the poem. G., who scored very high on the creative scale, used the verbal symbols also, but his understanding of the concepts was excellent, and the verbal symbols were not separate from the concepts that they represented. Thus he was able to manipulate the concepts very efficiently and very rewardingly.

Heinke suggests that new conjectures result from old ones by procedures which he organizes into three categories: i) variation of the data, ii) variation of the conclusion, and iii) others. The "others" category includes the converse, the inverse, the contrapositive, and the dual principle of a given statement.

Variation in data may necessitate a change in conclusion or may just supply new data to support the same conclusion. Similarly the change in conclusion may or may not effect a change in the data. The means by which variation in data or in conclusion may be obtained, as established by Heinke, were discussed in Chapter II, page 43. This list is presented again in Table 24, and at this stage, each method is illustrated by an example from the collection of student responses obtained in the present study.

Most of the variations will originate from the hypotheses originally provided in the constructed problem-situations, a fact which may be assumed in the following discussion unless otherwise specified. The original statements are:

i) A series of pentagons cannot cover a flat surface without leaving a gap unless the pentagons overlap.

ii) The area of a triangle is directly related to its perimeter.

The "other" category includes the following techniques for changing one conjecture into another.

i) Converse. A common occurrence was for students to use the concept of a plane interchangeably with the concept of a square. Thus a converse of the original statement would read as follows: "A series of squares would not cover a pentagon." The converse of one aspect of the triangle conjecture was as follows:

> "If perimeter increases would the area increase?"

The following two statements were also made by a student:

> "The area is proportional to the altitude."
> "The altitude is directly proportional to the area."

If proportional implies directly proportional as it seemed to do for this student, then the two statements are also converses.

TABLE 24

STUDENT RESPONSES ACCORDING TO HEINKE'S CLASSIFICATION FOR VARIATION OF DATA

VARIATION BY		STUDENT RESPONSE
Addition:	of data	A series of pentagons along with some triangles should be able to cover a flat surface.
	to conclusion	A series of pentagons would not cover a perfect square unless you added more surface area or unless the shapes overlapped greatly.

Note that this latter hypothesis also makes a substitution in the conclusion by substituting a more specific square for the more general plane. It was noticed that many students seemed to treat these two ideas as being equivalent.

VARIATION BY		STUDENT RESPONSE
Substitution of one specific piece of numerical information for another:		
	of data	If the sides of a pentagon are all 6 inches would the area remain the same if you had three sides, 6 inches, one side, 3 inches, and one side, 9 inches?
	to conclusion	Compare the change that one student made in the following sequence: Pentagons can be divided into three triangles which are not equal. Pentagons can be divided into four triangles, three of which are equal, one unequal.
Generalization: Substitution of numerical information by non-numerical information:		
	of data	This occurred in the triangle question when many students calculated areas of the triangles, perimeters of the triangles, then drew generalizations for the relationship.

Table 24 (Continued)

STUDENT RESPONSES ACCORDING TO HEINKE'S CLASSIFICATION FOR VARIATION OF DATA

VARIATION BY	STUDENT RESPONSE
to conclusion	Similarly G. M. played around with numerical relationships between the sides of the repeating pentagon series obtained by drawing the diagonals. The result--The area of one pentagon will be proportional to another. A second idea did not reach completion in the given time--that of finding the relationship between the area of the pentagon and the area of the circumscribed circle.
INDUCTION: Substitution of non-numerical information by numerical information:	
of data	When the perimeter is changed by one, the area is changed by two units.
to conclusion	Note that this hypothesis is illustrative of a change in both the data and the conclusion.
SUBSTITUTION OF ONE RELATION FOR ANOTHER:	
of data	Compare these two hypotheses given by one student. The conclusion remains the same even though the data changes. "The area of any triangles formed by the two sides of the pentagon and any line that joins two vertices of the pentagon will be the same." "The area of any triangles formed by the sides of a pentagon and any two lines that join two vertices of a pentagon will be the same."
to conclusion	"When you divide the pentagon into triangles by the altitudes are the triangles the same size?" "When you divide the pentagon into

Table 24 (Continued)

STUDENT RESPONSES ACCORDING TO HEINKE'S CLASSIFICATION FOR VARIATION OF DATA

VARIATION BY		STUDENT RESPONSE
		triangles by the altitude are the areas the same?"
SUBSTITUTION which results in a change of the figure:		
	of data	A series of triangles can cover a plane.
	to conclusion	A series of pentagons cannot form a circle. A pentagon cut in half makes a trapezoid. If you put one pentagon on top of another, then you get a ten-sided figure.

ii) Inverse. eg. A series of pentagons can cover a flat surface.

iii) Contrapositive. The contrapositive of the pentagon hypothesis would read: If the surface is not a plane, then a mosaic of pentagons is possible. The following equivalent statements made by the students:

A series of pentagons can cover a rounded surface.
A series of pentagons can cover a pointed surface.

iv) Dual Principle. None of the given hypotheses seemed to fit into this category. Notice that for the hypotheses dealing with area and perimeter the dual principle is equivalent to the converse.

The type of response most prevalent depended upon the information and the conjecture given by the author. The variety of the type of response could be increased by pointing out to the students the way in which responses could be made. Most of the students indicated that they had never worked on problems such as these; a few suggested that these problems reminded them of science experiments and several compared the problems to Language Arts exercises.

The students collectively produced at least one response for each of Heinke's categories except that of dual principle. Most of the responses were variations of the condition rather than of the conclusion. The most popular way of establishing new conjectures was by variation of the specifics involved or by addition of data. The more complex, the converse, the contrapositive, the dual principle, required a more adequate separation of variables than most of these students were capable of doing.

Case Studies

The following discussion is presented in four sections. The first is a report of some of the ideas volunteered by four of the more "creative" individuals. The section following the report attempts to identify five levels of responses made by the students. The levels are exemplified by the individuals in the case studies, and by responses made by two other individuals. The third section summarizes the student responses to the question "How do you make a conjecture?", thus indicating the extent to which the subjects were consciously aware of the strategies they used to conjecture. The last section reports on some of the sequence of ideas which occurred in the minds of students as they produced the hypotheses. Since these sequences seemed reminiscent of Vygotsky's ideas, a summary of this author's theory is included.

The first two cases reported are of two individuals who worked in distinctly different ways and achieved different products. Both, however, may be classified as creative in the defined sense.

G. (14-6, 113, 131),[5] a boy who was very active in scientific and mathematical recreational study, used some very systematic and efficient mathematical thinking in his attack on the given problems. He treated the set of triangles as a sequence and dealt with the abstract concepts of area and perimeter, rather than with the specific numerical facts, when making his conjectures. He was more concerned with these than with the specific aspects given in the diagram and as a result, did not, as had most of his peers, check out all the details and facts present in the diagram. For example, he had not at the end of his initial fifteen minutes calculated the specific perimeter of the triangles.

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[5] (Age, Lorge-Thorndike verbal score, Lorge-Thorndike non-verbal score).

Preoccupation with the abstractions (area, perimeter, altitude, classifications) was characteristic of G.'s work throughout the experiment. This familiarity and confidence of the ideas which extended beyond any of the specifics given enabled G. not only to make more interesting conjectures but to give detailed proofs for the verifying questions and to classify the answers to the sensitivity questions in a systematic and orderly fashion. (The sensitivity question will be discussed in detail later.)

G.'s greater knowledge and facility with mathematical ideas enabled him to work at a much higher level than some of his colleagues. G. did not check his conjectures thoroughly, although he did "think about them partially to see if they made sense". He seemed capable of eliminating the more trivial ideas, and certainly was not reduced to just stating facts about the diagram.

G.'s firm grip on the concepts enabled him to manipulate the words and symbols to achieve his ends. In the triangle question, he examined the sequence of triangles, their similarities, and then quickly separated out the variables. Area and perimeter were suggested by the statements; altitude was suggested to him by the sequence, (shape of the triangle depended on the height). He then consciously focused on one variable, and compared that variable with the others in as many ways as possible. He then focused on a second variable and combined it into as many relations as possible.

It seems that G. had a fairly clear notion of altitude. One of the most common difficulties for students in the sample was that of separating the altitude from the length of a side in a polygon. This difficulty revealed itself distinctly in the first verifying question and formed one of the bases for the categorization of responses for that process.

The responses made by G. to the triangle question were:

1. The area of a triangle increases as the altitude increases.
2. The altitude of a triangle is directly proportional to the perimeter.
3. The altitude of a triangle is directly proportional to the area.
4. The areas of a triangle is proportional to its altitude.
5. The sides are related to each other, determined by the angles of the triangle.
6. Any pair of angles determines the third angle.
7. The altitude of a triangle is determined by any pair of angles.

G. had an abundance of ideas, and the number of ideas grew as the time increased. Many students would run out of ideas; for G. each idea multiplied. The following responses were made after the initial discussion and were not scored.

8. The areas of the two triangles that are formed by the meridian are equal.
9. The decreasing or increasing of the sides of a triangle at a proportional rate will affect the area of the triangle but not the angles.
10. The sum of the base and altitude is equal to the sum of the other two sides.
11. The sum of the two equivalent triangles placed so their base form a mutual diagonal will form a parallelogram.

The ideas although related do not repeat but proceed from one area to another, relating as many fields as possible.

When responding to the pentagon question, G.'s ability to consciously vary the conditions in one idea until he had examined all the combinations, and then to "look for some idea that I have not looked at before. . ." enabled him to produce some productive ideas. G. was also able to list and to separate some of the variables present in the pentagon, and, because of his ability to deal with these variables in the abstract, he was not bothered by the presentation of the pentagon without specific information. Many of the other individuals found the problem barren of stimuli, and were thus unable to relate to the data.

The following conjectures illustrate how he was able to use conscious variations similar to those identified by Heinke.

1. A series of pentagons can cover a flat surface without leaving gaps only if the flat surface represents a pentagon.
2. The area of any triangles formed by the sides of the pentagon and any line that joins two vertices of the pentagon will both be the same.
3. The area of any triangles formed by one side of the pentagon and any two lines that are joined at one vertex of the pentagon and its other endpoints by two other vertices will be the same.
4. The area of one pentagon will always be proportional to another.

The following conjectures were offered and not scored, since they occurred after the initial fifteen minute period.

5. Within any pentagon, when the vertices are all joined by lines, there will always be two sets of five congruent triangles and one pentagon in the center.
6. The apothem of the pentagon times the perimeter will result in the area of the pentagon.
7. Congruent triangles will be formed when vertices of the sides are joined by lines onto a common point in the center.
8. There can be a minimum of six pentagons that can be placed into one pentagon, not counting the fact that one pentagon can also be fit into one pentagon.

G.'s very deductive and logical approach, used in the formulation of all his conjectures, is indicated by this comment by him on his method of making conjectures about the pentagon situation:

> I thought that the conjectures about the area of the triangles were almost realized, so I thought about the area of something else, the lines I had drawn indicated a pentagon, so I wondered about the area of the new pentagon.

This approach is in total contrast to the one reported next.

C.'s (14-10, 142, 113) approach and attitude were totally different. At the end of the experiment, she very clearly stated that she had been able to distinguish between the open-ended questions as compared to those that were not. She preferred the open-ended ones because she preferred to ask questions rather than to look for a definite solution, as in that way she could choose those ideas which appealed to her and then follow up those ideas.

C. proceeded by asking questions such as "What would happen if I did this. .?" For example her initial reaction to the pentagon was "If I could fold in the angles of the pentagon, and if it did make a star, if it would be along the same lines as the diagonals?" She then began to wonder about the relationship between the original pentagon and the one that she would form? This approach was unusual; she started with the angles and an artistic, craft-like approach to the diagram, then realized that the edges of the angles might lie along the diagonals to form a second pentagon. This type of approach rather than a logical one was noted throughout the experiment but especially in her response to the first sensitivity question. (See page 277) Although the diagrammatic approach to the pentagon was common, and the use of the figure to make designs was noted, the gap from the artistic notion to a mental questioning about the properties or the relationships specifically mathematical was not often bridged.

How does C. form conjectures? The following excerpts from the taped responses indicate some of C.'s own analysis of the procedures that she used.

> A: I have to think of one idea. Then the others just come, mainly because they seem to relate, sometimes I know that they will relate, other times I don't.
>
> Q: What makes the second ideas come from the first one?
>
> A: Well, they are just variations of the same idea. . . . And then as soon as I look at those and I know its a group of one thing, that a group of shapes inside this, then I think about something else that is a group of another thing.
>
> Q: Do you try to formulate some definite ideas about what will happen as a result of some of your ideas?
>
> A: I don't usually try to find the answers, I just ask the questions.
> . . . and if I had a lot of time, I think about all the things I was thinking about first and then, I'd take the ones I couldn't stop thinking about, and then I'd work those ones out.

The contrast between G. and C. may again be noted. G. has total cognizance of the final result and isolates one possibility at a time which he investigates; C. follows up the idea which appeals to her at the time, and is reluctant to commit herself to one possibility.

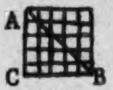
Because of her whimsical approach C. made fewer well-defined conjectures than did G. Some of her conjectures are presented below.

These had to be clarified to some extent because her ideas were fragments of sentences and thoughts.

1. A series of pentagons would cover a pointed surface?
2. A series of pentagons would cover a rounded surface?
3. What would you get if you folded in the angles--how many lines, triangles on sides, pentagon in the middle. . .
4. Pentagon in the middle is not in the same position as original.
5. Are the lines diagonals?

The following conjecture was made after or during the oral interview and so was not scored, but it is a result of her previous, rather disjoint conjectures.

6. What is the relationship between the size of the original figure and that of the pentagon formed by the diagonals?

C. was also able to separate out assumptions. For example, she was able to question whether her folded angles and the diagonals were one and the same. She was also able to distinguish the variables which were present in the second question. Her first thought on the question was "Instead of area varying as perimeter it varies as altitude." This led her in a direction not discussed by anyone else, the length of the slant lines and their relationship to the other two sides. Whereas other students often made note of the slant line, none had tried to compare that length to the altitude. She seemed fascinated by how one would estimate the length of these lines. One could not count squares to find the measure as one could in the case of altitude and base. This fact provides difficulty for most junior high school students. Although they may realize that distance AB is greater than distance AC, they still try to find the length of a slant line by counting the number of squares that the line cuts. 

Various ideas passed through her mind--no right angles, moving triangles together, straightening the triangles, the shape, and finally wondering what effect angle size and shape had on slant length especially if she straightened the triangle into a right angle.

Her concern seemed to be with general concepts rather than with specific instances. C. had been influenced by some reading on perception (the psychological studies on the ink-blots) and was playing around with shapes and impressions that were perceptual.

C. and G. seemed equally capable but in a different way. Both individuals scored high on the constructed tests, and both supplied responses that are interesting and worthy of note. G. however was considered an excellent student by the school and had channelled his energies into doing well at school as well as pursuing his academic interests outside of the classroom. He had immigrated to Canada when in elementary school and thus his verbal intelligence score at this point was not a true indication of his ability. C. on the other hand, was doing poorly in classroom work. The counsellor, although recognizing C.'s potential, voiced some fear as to her achievement, and her future school career.

Although neither C. nor G. actually evaluated their ideas, they were able to select relevant paths. There were individuals in the sample, however, who tended to generate new ideas as a result of a somewhat more detailed examination of their initial ideas. Success with this type of approach depends upon how quickly one can analyze his ideas to see where they lead. The individual who has a lesser knowledge may spend more time examining initial ideas before realizing the most fruitful direction. GM.'s case illustrates how difficulty in computing specific instances may delay the formation of conjectures. In a setting with a time limit, this delay influences performance scores.

GM. (14-10, 140, 131) gave indications that he would formulate very strong hypotheses, but in the time allotment did not write down notable ideas. GM.'s approach was a very intuitive, basic one which did not depend upon memorized data. To establish area and perimeter relationships in the triangle question, he attempted to match pieces of triangles together, utilizing a very basic notion of area.

Although GM.'s method was admirable, it resulted in an incomplete conjecture, and thus, in the limited time available, was less efficient than the method used by G. He did decide that area was dependent upon perimeter, but did not realize the extent of the relationship. However this examination did lead to some ideas as to the relation between angle size and side size. The idea unfortunately was not questioned closely by the researcher during the oral interview. It was only upon reflection and further experience with GM. that the possible strength of his idea was realized. GM. did not express his ideas well, and it was only during the course of the experiment that it was realized that conjectures which at first were considered trivial or incorrect were potentially strong. GM. only wrote down the more obvious aspects of his ideas or the ones he had checked out totally.

GM. proceeded in a similar manner while working on the pentagon question working from basic ideas, getting a total picture of what was available in the figure. Again he did not write down notable hypotheses, but when questioned voiced two concerns, that of the relationship between respective pentagons, and later that between the pentagon and the circumscribed circle.

His written hypotheses were:

1. If the pentagon is rotated around one point until one revolution is made, a circle is formed.
2. A pentagon with congruent segments can be inscribed in a circle.
3. One can obtain the same pentagon in perspective increases by forming star shapes.

These ideas do not indicate the depth of the ideas expressed during the oral interview.

Q: Can you tell me some of your present thoughts?

A: Well right now I'm seeing the relationships between different sizes: I'm going to get the area of this (pointing to the original pentagon) and the area of that (pointing to the smaller enclosed pentagon), see if the increase in that uh, check the area of this one with that one and then the area of this one with that one, (GM. indicated the series of pentagons, two at a time, choosing the two sequential ones)

Q: What made you think of that?

A: First of all I was working with, my first one was circular shapes, and I was trying to find the center, so if you placed a compass point . . ., so I was trying to find the center point, then I found that I made another pentagon in the center here that was just sort of inverted . . . So I then did the same thing with that and so on till I could relatively approximate the center point.

Q: And that gave you the idea of finding the area of the pentagons?

. . .

Q: What gave you the idea of the circular shapes?

A: In Math. Regular shapes can be inscribed in circles. So theoretically then if you just move one point every time, from this side to this side it would be enough to give you a circle. And then you could have circles inside, or touching the sides of the pentagon . . . .

(Some further discussion of the same)

Q: How does one idea come from another? Your conjectures just lead from one to another. How did that happen?

A: First of all I, took this one first just thinking of this, seeing if it was right or wrong, and just as I'm working an idea I hadn't though of before comes up while working, then I just leave that and go on to a different area.

First of all I was thinking of all pentagons in general. Like I was thinking of this one here, but not thinking of it as a regular pentagon but just a pentagon in general, and then as I got something for that, I took a closer look at this and found it was a regular pentagon so that I could stay on the same idea as this but modify it somewhat.

Q: Have you done things like this in class?

A: No, I don't think so. . . .

. . . (Discussion on the given hypothesis, GM. started to explain why the mosaic would not result, at first supposed that this would depend on whether the number of sides was odd or even.)

Q: What made you think of comparing areas?

A: Well mainly because it describes the size better than would maybe perimeter, although the two are closely related.

Q: What made you think of using numerical data, like area, for example?

A: Probably because its that I like working more with numbers than with shapes. I like to find abstract differences, not something that you can actually see, but sort of abstract relationships between this and something else.

Q: What suggestions would you give someone having difficulties in making conjectures?

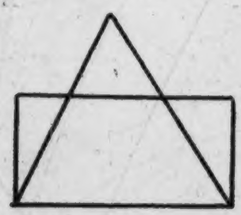
A: First of all I'd ask them to take a look at this first one, did you try it to see if it'd work out and if it doesn't work out, why doesn't it work out? And if it doesn't work out for this will it work out for other shapes? And then how can you use this to form other figures?

If you join every point with every other point, just seeing what happens, and what you can do with the result of what you have here.

In answers to the same question when dealing with the triangles GM. suggested the following:

i) Look for similarities between the triangles.
ii) Look for interrelationships that made the triangles a set.
iii) Look for the differences between the triangles.
iv) Vary the shapes of the polygons.
v) Check other shapes and compare them with the triangles in size.

After working with the triangle question, GM. did not have time, due to class changes, to discuss his second hypothesis--"the perimeter of a given triangle equals the perimeter of a rectangle two units smaller". This conjecture is again unclear in meaning but assuming he means size as in area, his idea seems to stem from examination of the triangle and its rectangle as shown in the diagram. This type of diagram was part of his exploration shown in his rough work.



The hypothesis, relating the rectangle and the triangle by a sum, rather than a product, occurred even though GM., a good student, knew that the area of the triangle is given by the formula $A = \frac{1}{2}bh$. This is just another example of the fact that concepts are formed in partial stages, the actual concept growing as new experience is brought to bear on the individual. It is difficult for an outside individual to tell just how much the concept as realized by a child matches the concept that is specified by mathematical authority.

CL. (14-7, 127, 124) exemplifies another approach to solving problems. He claimed, during the solution for cutting the square in half, that he looked for differences, for different solutions in order to achieve ideas. As have the other three individuals reported above, CL. has a clear understanding of what made a significant conjecture, and was not content to list only observed facts. His work with the triangle question illustrates the effectiveness with which he generated new ideas by evaluating given ones. This is in contrast with his work with the pentagon. Here he was unable to "get started" on an idea that generated new ones. As a result, although his numerical examinations and explorations with the sequence of triangles were in light of an abstract relationship, much in the same way as were GM.'s, his numerical work with the pentagon deteriorated into a description of the specific polygon. This was totally frustrating to him, but although he needed help in locating a relationship to investigate, and even though, several ideas for investigation were suggested, he appeared unable to submerge himself into any of them. In contrast, he was so interested in the triangle problem that he took it home to investigate further at his leisure.

CL. seemed at the point at which further knowledge was necessary. From a teaching standpoint, this may have meant a reading assignment, a discussion directed by someone more knowledgeable than himself, a lecture, or some other directed stimuli. CL. did not see the importance of the implication of drawing the star; he did not see any value in comparing line segments, and did not fully appreciate the significance of his own list of polygons that could make a mosaic. Given more time, this last idea may have been promoted by further questioning. The significance of an individual's background knowledge seems to have been emphasized by this case. CL. did not pursue any of the above ideas because he did not realize or appreciate the types of relationships they might have created. The other three students all sensed these possibilities. On the other hand, CL.'s experience with triangles seemed to have enabled him to pursue some potentially fruitful ideas. It is also possible that the more relaxed individual pursues these avenues even though he may lack knowledge and that someone less concerned than CL. with achieving a good answer may be more willing to take the chance to investigate ideas without fully foreseeing the results.

The English teacher commented that CL. set very high standards for himself, and that he was never confident of meeting them. For example, he would write a "perfect essay" and include the comment at the bottom: "At least I try very hard."

The following excerpt from CL.'s comments on the triangle question illustrates the high quality of work produced and thereby emphasizes the gap found between production on the two questions. It also shows the interaction present between conjecture and verification which enabled CL. to proceed through the problem and led him eventually to his self-assigned math project for homework. Note also that CL.'s ideas are very closely tied in to the relationships he would have had discussed in class.

> Both sets of measures add up to 18. The lower right triangle does not have the same perimeter as the others if this is true--yes it does. The greater the angle, greater than 90 degrees, gets on the upper right and lower left, with the base remaining the same, the larger the side opposite gets, and the smaller the side adjacent to it becomes. The greater the angle greater than 90 is, the less the altitude is, and the connected length of the line opposite the angle increases in order to join with the line adjacent to the large angle. The area of the triangles should be the same according to earlier thoughts, but it isn't. Therefore I conclude altitude must have something to do with it. I find that a and d have the same area and the same altitude (He is approximating and is slightly off, however the idea is correct.) and base so, if the base and the altitude of a triangle are the same, the areas are the same, regardless of measurement of angles and other ideas.

Next day he brought a page of investigations which centered around the concerns he had voiced above.

> Could there be some way triangles' measures are related?
> Like pi for a circle.
> Do squares and rectangles have this, maybe like a circle?

His comment near the end of the examination--"It probably couldn't do anything with sides and area--maybe cross distance times something gives area."

CL. did have an interesting idea about the pentagon which he discarded after evaluating one counter-example. He suggested that every pentagon might have a total interior angle sum of five hundred and forty. He took specific measurements of one irregular pentagon, found the sum to be greater than five hundred and forty (551) and discarded the idea. In this case CL. was unable to use an incorrect idea to generate further hypotheses. The confidence necessary to do this may require further experience with an area previously unexplored.

Levels of Response

The above report on the case studies indicates that the students in the sample were aware that data can be varied, and they intuitively used the procedures suggested by Heinke. The information in the report also suggests that the students used three levels of attack on the problem-situations. The top level, exemplified by G. and C. involved the examination of the concepts and the variables in the abstract sense. A second level involved the use of numbers in an attempt to develop abstract relationships. This level is best illustrated by the work done by GM. and the work on the triangles by CL. A third level occurred when the attempt to find abstract relationships failed, and the result was a listing of numerical data and observation without the presence of an overriding hypothesis. This was illustrated dramatically by CL.'s experience with the pentagon.

Responses were also made on two lower levels. In one, the individuals were aware that additional abstract relationships were required, but the dominant need was to establish the correctness of the given statement by using some numerical calculation. Some students were particularly fascinated by the fact that perimeter and area were not totally dependent upon one another; others established the fact and questioned it no longer. Often these latter individuals were unable to proceed with ideas of their own.

M. (14-8, 143, 130) for example, found the area, found the perimeter, saw very clearly the idea that

> Triangles with the same perimeter do not necessarily have the same area.

and that

> If one side of the triangle was changed, then the altitude would change also. If the base was increased, then the altitude would also decrease, (that is, if the sides remained the same length) and vice-verse.

From there she was unable to proceed, even after the discussion.

M.'s paper showed very little exploratory work. She had apparently calculated the areas mentally and had investigated the suggested relationship very efficiently. She had not, however, investigated any other combinations or possibilities with the words or variables or the diagram.

JH. (14-6, 119, 130) was less efficient and in character seemed less concerned with obtaining the one correct answer than was M., but also had difficulty in extending her ideas beyond the suggested relationship. JH. constructed the following conjectures:

> i) When the bases are the same, the higher the altitude the shorter the third side.
> ii) There are not right angles in these triangles.
> iii) The bases are all six.
> iv) The perimeter is eighteen.
> v) If the base remains the same the higher the altitude the larger the area.

She could not continue further. The conjectures are all observational and converge towards a final conclusion, rather than to the opening up of further questioning as had the conjectures of G. or C., or of GM. This difficulty was common with the responses for this question. It seemed to present a problem which required justification, and at that point, either due to lack of time or to a set, imposed by the given conjecture, created a hurdle.

There were many responses of still lesser quality than those given above. Some were numerical, but calculations were either incomplete or incorrect because of the lack of knowledge or manipulative skill possessed by the student. Sometimes relationships other than that of area, such as additive or pythagorean relationships between measures of sides were attempted and left incomplete due to the time and the lack of arithmetical skill. This was unfortunate, but occurred in several cases even though a pre-test and a pre-teaching session had taken place. These few individuals were not able to retain and apply the concepts and were the lower mathematical achievers.

Perhaps the least creative responses were those which were strictly a recording of observational facts. Note for example JC. (14-5, 112, 110):

i) They are all three-sided.
ii) The perimeter of triangle one is 18.
iii) The perimeter of triangle two is 18.
iv) The perimeter of triangle three is 18.
v) The perimeter of triangle four is 18.
vi) Triangles 1, 2, 3, 4 all have the same perimeter.
vii) The area of triangle one is 30.

(This was incorrect. He forgot to divide by two.)

A somewhat higher level response than given by JC. was that of giving abstract relationships without reference to the actual numerical data, but by manipulation of the given conjecture. This differed from the type of manipulation by individuals such as G. in that it was very closely defined by the given statement, or some one statement of the student's. Just as some individuals were bound by the given diagram, these individuals were bound by the written word. This may indicate that although their intellectual (verbal) level had progressed to a high stage, the set or inhibition had not released their perspective. Such an example was ES. His hypotheses for the pentagon follow:

i) If enough pentagonal shapes were used in a certain formation it would be possible to cover a flat surface. (restatement of original)
ii) It is impossible to create a pentagonal shape out of any combinations of squares.
iii) A pentagon can be composed of ten isosceles triangles, and a smaller interior pentagon.
iv) It would be impossible to create a pentagon out of any number of smaller pentagons.

His hypotheses, at face value, were very fine reorganizations of a few variables. He, however, was unable to ask the same questions about size relationships, about why some formations were possible and how the various triangles related, that some of his colleagues posed. The difference was not evident in the written work. The difference was that C., G., and GM. were using this type of reorganization as stepping stones to further questions; ES. was using them as a final result.

It is suggested that the above examples represent some sort of hierarchy in creative thought. Although it is difficult and dangerous to say that one of the above individuals is more creative than another, the abstraction of the responses made in this one specific experiment, suggests the tentative classification of responses into these 5 levels.

A. The use of abstract concepts as variables in combinations to result in new relationships.

B. The use of numerical examples to establish and to generalize some new relationships.

C1. The use of numerical data to investigate a certain idea followed by a sudden inability to suggest relationships.

C2. The use of the verbal ideas given to generate hypotheses, but an inability to develop new relationships, once all combinations of verbal ideas are considered.

D. The calculation of some specifics, which are recorded as observations about the given data. No relationships are sought.

E. Recording of the given or observable data with no further calculation, no supposed relationships.

Individuals in the last category were able to realize perimeter, however, did not investigate the more difficult concepts of area, and altitude, even by intuitive means such as the counting of squares in Conjecturing II.

Pupil Awareness of Responses Made

Many individuals were partially aware of the procedures that they used in making their conjectures. Here are some of the replies to the question: "How do you make a conjecture?"

1. Find an interesting idea and develop it by

examining loopholes in the conjecture.

2. Figure out everything, then see how to change and to establish relationships.
3. Look for similarities, interrelationships, differences, and some ways to vary the given shapes.
4. Look for similarities, differences; Use one variable in the sequence at one time.
5. Decide if the given conjecture is true or false. If false, try to establish conditions under which it would be true.
6. Invert or negate a conjecture. Vary a shape.
7. Look for similarities.
8. Look for similarities and differences.
9. Use a basic idea; check it if it is right; if right, look at all the ideas which inspired it and attempt to get related ideas from these ideas; if incorrect, look at the reasons why it is incorrect.
10. One should consider an idea even if it sounds irrelevant and maybe you can do something with this problem.
11. I think you just have to sit there and look at it long enough and get ideas and be unrealistic if you have to, to get yourself going; so I figure that about the best way is to dream a little and then you'll start to get to the right thing. Substitute one idea for another, for example, substituting the pentagon as an alternative for the square as a unit of measure. (This individual, although considered a weak student, indicated that she liked to write poetry at home, and compared her thought processes here to the ones she used for getting ideas for her writing.)
12. Take a small example and build it up as far as it would go. It's finally like a chain reaction.
13. Conjectures are a result of adding lines or ideas to the figure, then trying to see the relationships between the ideas in the diagram. Relationships turned out to be similarities, differences, and inversions although the inversion was unconscious.
14. The first day I came in here, I didn't know what you were talking about, but when you're doing it like this you get used to picking anything and not being afraid of whether its right or wrong, just trying different things, just seeing what will happen if you increase something a little more or something like that--just stretching it around in your mind and taking it out to see if it works. . .its just trying different things and you're not afraid if you're wrong about it.

15. Consider the following things: thinking of all the possibilities, thinking of all relationships between the possibilities, thinking of why certain relationships don't hold, thinking through some problem from these questions, "This way you could think out a problem and then get hypotheses out of that."

The above ideas reoccurred during the discussions with the individuals. Although these responses came from those individuals who were aware of the techniques that they were using, there were students who used some of these ideas without realizing that they were doing so. Some of these were surprised pleasantly when the experimenter indicated that maybe they were thinking this way.

Sequences of Ideas

The students often generated new hypotheses by associating an idea from a previous hypothesis to a new area and thus another idea or relationship. Some of these sequences were revealed in the oral interviews. These may relate to some of Vygotsky's views on concept formation.

Vygotsky claims that the child's ascent to concept formation occurs along a continuum which can be described in three stages, each divided into several substages. The first step toward concept formation by the child is a gathering together of a number of "disparate objects grouped without any basis and which reveal a diffuse undirected extension of the meaning of the artificial word to inherently unrelated objects linked by chance in the child's perception." (Vygotsky, p. 59)

Three substages describe this syncretic organization of objects:

i) The trial and error stage. The group representing the given word (concept) is created at random and each object added is a mere guess or trial; it is replaced by another object when the guess is proven wrong.

ii) Syncretic organization of the child's visual field-- The group of objects which represent the required concept is formed as a result of a more or less complex relationship between object in the child's immediate perception field, for example the objects contiguity in space or time.

iii) A combination of objects taken from syncretic heaps previously formed as in ii) or i).

The second stage Vygotsky calls Thinking in Complexes, a state during which individual objects are united in the child's mind not only by his subjective impressions but also by bonds actually existing between these objects. The bonds which exist between the components of the complex are concrete and factual. Vygotsky compares this type of organization to a family organized by their last name. A person belongs to the Petrov family not because of any logical or abstract relationship but because of a factual one.

Thinking in Complexes is subdivided into five stages. The associative type is based on any bond which may exist between an already existent object in the class representing the concept and other objects. In Vygotsky's experiment, children were to group a set of logic blocks according to some set concept predetermined by the experimenter but unknown to the children. A sample block was given the child who formed the remainder of the set by an inconsistent (adult judgement) set of rules. For example, subsequent blocks may be chosen as follows: B_2 same size as B_1, B_3 same color as B_1, B_4 same shape as B_1 and so on.

In the second type "the child would pick out objects differing from the sample in color, or in form, or in size, or in some other characteristic. He did not pick them at random; he chose them because they contrasted and complemented the one attribute of the sample which he took to be the basis of grouping". (p. 63)

The chain complex is a more dynamic linking of objects into a class, the meaning being carried from one link to the next. For example, the following blocks may go into one class: yellow triangle, yellow triangle two, blue triangle, red triangle, red square, red rectangle, red circle, yellow circle, blue circle.

The criteria by which the class is formed is changing. This indicates that each block enters the class as an individual rather than a carrier of a common trait. The single trait has not been abstracted, the attribute is a perceptually concrete and factual thing.

The last stage in complex thinking, the highest stage previous to formal concept formation, is the pseudo-concept. In this case the same class results as would from an abstract organization or concept. Vygotsky cites observations by Hanfmann and Kasanin which indicates that the classification is guided by concrete likeness and is a perceptual bond rather than an abstract analysis and classification. Upon presentation with a counter-example, the pseudo-classifier can only discard the counter example;

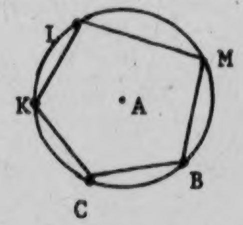
the classifier who has actually formed the abstract concept can reorganize his total set to meet the additional objections of the counterexample.

Pseudo-concepts often appear as a result of training or of imitation of some observation the child made of an adult performing a similar task. Thus the actual distinction between the pseudo-concept and concept stage is very difficult to discern until the concept must be applied in a new situation.

The sequence of ideas which passed through W.'s mind as he produced his conjectures might be compared to Vygotsky's second stage of complex thinking. The concept of shape pervades his ideas and provides a link to chain ideas as they pass from one specific idea to the next, however the bond changes from one pair of ideas to the next pair of ideas.

W.'s conjectures were:

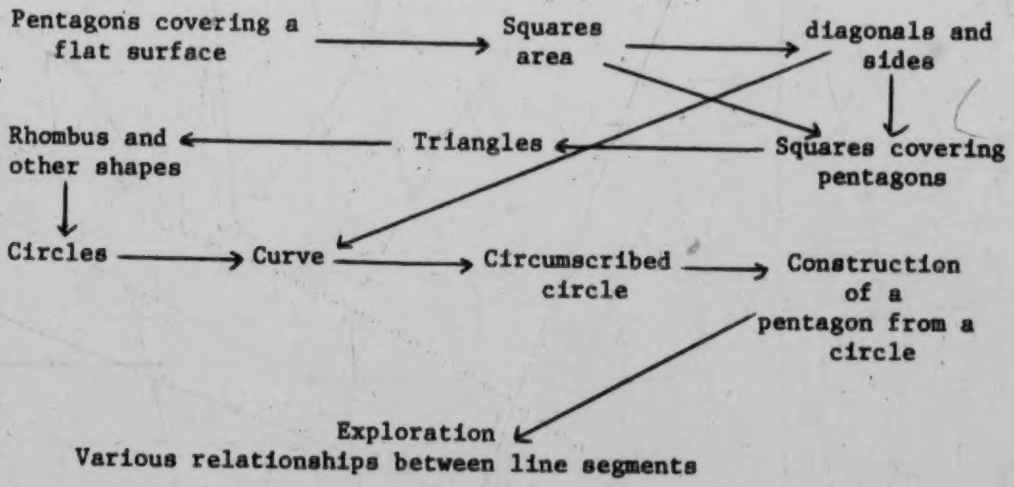
i) Is it possible to arrange the triangles and the pentagon from the pentagon into a square?
ii) A circle with center A, radius AB = AC, would also pass through points K, L.



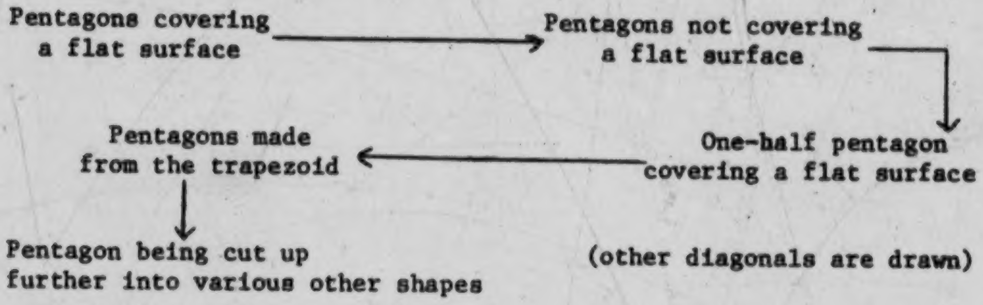
(The second hypothesis came after the oral discussion, and although it was not scored for this reason, it was, in the investigator's opinion, W.'s original thought.)

W.'s initial hypothesis seemed to be related to the given idea, that of forming a mosaic from pentagons, however he did not seem to realize this relationship until it was drawn to his attention. He had started by testing out the given hypothesis, but had tried to establish the shape of polygons resulting when pentagons are placed together. The idea of covering a flat surface related to the question of area. (He seemed to treat flat surface as a finite area.) The idea of area was at this stage, only a fleeting idea, but the idea of area in terms of a square, combined with the diagonals and the sides of the pentagon led him to consider triangles, then the rhombus, then other shapes including the circle. The idea of circles was temporarily discarded by W. but the idea of connecting vertices suggested the notion of a curve and again a circle, this time a circle that was the circumcircle for the pentagon. This raised the possibility that a pentagon might be constructed from the circle. After the discussion and the suggestion that the diagonals may have a measure and that this measure may be of interest, W., in searching for a center, discovered the second idea.

The ideas as they change can be summarized by the diagram:


Pentagons covering a flat surface
Squares area
diagonals and sides
Rhombus and other shapes
Triangles
Squares covering pentagons
Circles
Curve
Circumscribed circle
Construction of a pentagon from a circle
Exploration
Various relationships between line segments

The same initial idea, that of pentagons covering a flat surface, gave rise to an alternate sequence by RT. She followed the ideas this way.


Pentagons covering a flat surface
Pentagons not covering a flat surface
One-half pentagon covering a flat surface
(other diagonals are drawn)
Pentagons made from the trapezoid
Pentagon being cut up further into various other shapes

W.'s ideas on area and flat surface also show some of the characteristics of a pseudo-concept. The idea of a plane was considered in terms of a specific shape, mostly that of the square. The thinking about this concept was then guided by concrete or perceptual relationships not by abstract analysis.

In RT.'s case, the bond between pairs of ideas was changing. Although no conclusions can be based on the limited data, the sequences in thinking suggest that it might be possible to classify responses in reference to the stage of concept formation they indicate for a specific individual. This type of classification would emphasize the way in which a response was formulated rather than the response itself. Further research is necessary before the plausibility of this approach is examined.

The above two sections have dealt with student responses to the conjecturing situations. There has been an attempt to classify the responses according to Heinke's organization of variation, and according to the level of response and approach used by the student.

Some indication has also been suggested for a classification of sequence of ideas in terms of the theory of concept formation as suggested by Vygotsky. The following section will consider some of the mathematical possibilities in the constructed conjecturing situations. Reference will also be made to the possibilities for the problems as teaching situations.

The Potential in the Conjecturing Problem Situations

The Pentagon.

The pentagon can provide many hours of exploration for anyone at any level of knowledge. Children at the grade nine level can raise many interesting questions and conjectures, as can be seen from the list included in Table 25. This table presents some of the student responses in the first column; the area of study which is related to the student conjecture and the suggestion of a reference which is directly applicable to the area of study in the second column.

Some of the given student responses listed in Column I are correct, some are incorrect, but most of them can serve the discerning classroom teacher as valuable starting points for mathematical exploration and discussion. For example the given conjecture evoked the following responses from the students:

i) Could a series of pentagons cover another pentagon?
ii) Could a series of squares cover a flat surface?
  a pentagon?
iii) Could a series of triangles cover a pentagon?
  any flat surface?

TABLE 25

STUDENT RESPONSES TO THE PENTAGON AS A SOURCE OF MATHEMATICAL STUDY

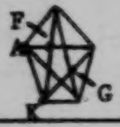
SOME HYPOTHESES	
i) A series of pentagons can cover another pentagon	i) Study of mosaics
ii) A series of squares can cover a flat surface; a pentagon.	ii) Number of degrees in a vertex angle of a polygon. Stover - Mosaics. eg. of theorem: There are most thirty mosaics.
iii) A series of triangles can cover a flat surface; a pentagon.	iii) Study of finite mosaics (vertex angle < 360) polyhedra formed from this arrangement. (Could lead to study of solid figures) Problem of packing (Gardiner - Sc. American)
iv) A series of pentagons can cover a rounded surface.	iv) An important lemma in the proof of the golden section; that numbers which are irrational exist, an example of Fibonnacci series from the relationship between the diagonal 1 side of a square $d(d - s) = s^2$ See historical accounts of Pythagorus and his followers. eg. Ways of Thought of Great Mathematicians, Meschkowski.
v) When you lay a series of ⬠'s side by side, you will have to travel a curved path in order to reach your starting point.	
vi) If a compass point is put on point K and a radius AK is chosen for the circle, the arc cuts the diagonals at F, G.  	

TABLE 25 (Continued)

STUDENT RESPONSES TO THE PENTAGON AS A SOURCE OF MATHEMATICAL STUDY

SOME HYPOTHESES	
vii) A pentagon can be constructed in a circle. →	- Study of constructions, study of properties of polygons,
viii) The area can be calculated by calculating the area of 3Δ's.	- golden triangle Border of Mathematics
ix) The area of a pentagon is equal to the 5 's found at the center.	(1) CALCULATION OF AREAS.
x) If the sides of a pentagon are all 6", would the area remain the same if you had 3 sides 6", 1 side 3", and one side 9"?	(2) AREAS OF POLYGONS DEPENDENT UPON ANGLE SIZE, CONSTANT P.
xi) Is it possible to use a pentagon as a unit of area?	(3) CONGRUENT TRIANGLES
xii) The area of any two Δ's formed by the 2 sides of the pentagon and any other line that joins 2 vertices of the pentagon are the same.	(4) PROPORTION _ SIMILARITY OF POLYGONS
xiii) The area of any Δ's formed by the side of a pentagon and any two lines joining the respective vertices are the same.	
xiv) The area of one pentagon $\subset$ the area of another ⬟ pentagon.	
xv) What relationship exists between the area of a pentagon and the area of its circumscribed circle.	

One student followed up these ideas by establishing for himself that pentagons could not cover a plane since the measure of each angle was not a factor of 360°. The study of tesselations and mosaics on the plane may follow directly from such an examination resulting from the questions posed above.

The question of angle measure also relates with a second category of response popular with the children--that of subdividing the pentagon into sub-polygons, the most common being that of the triangle. The following statements are illustrations of this type of response.

i) The pentagon can be divided into three.
ii) You can make five triangles by joining the vertex with the center.
iii) The figure is made from five similar isosceles triangles.

Any of these responses may have resulted in another student response, "There are 360° in a pentagon." The algorithm for determining the sum of the measures of the interior angles for a pentagon and more generally for any polygon (Sum = $\frac{180\ (n - 2)}{n}$, where n is the number of sides) might follow from an examination in the classroom of the division concept combined with the concept of angle sum.

The concepts on mosaics can be extended into three dimensional space and the study of polyhedra by examining mosaics whose angle sum at a vertex is less than 360°. (Stover, 1966)[6]

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[6] Some theorems that are developed by Stover are the following.
i) No mosaic symbol may contain more than six numerals eg. (3,3,3,3,3,3) and the only one containing six numerals is the one given above.
This would indicate that the triangle is the simplest polygon that may be used to construct a tesselation, also that a maximum of six equilateral triangles can be placed at one vertex.

ii) There are at most thirty mosaics.

Some conjectures given by the children which can generate this type of extension are the following:

i) A series of pentagons can cover a rounded surface.
ii) When you lay a series of pentagons side by side you will have to travel a curved path in order to reach your starting point.

A completely different direction may be pursued as a result of the pupil conjecture number vi) listed on Table 25. The pentagon encloses a number of special isosceles triangles. The 72°, 36°, 36° triangle which is regenerated under a fibonnacci curve, and thereby called the golden triangle results from the relationship which exists between the diagonal and the side of the pentagon as well as the proportional lengths which are obtained when the diagonals cut each other.

Hippasius (Meschkowski, 1964) constructed an infinite series of pentagons using the diagonal of an initial pentagon to establish the side of the subsequent pentagon. The relationships are shown in the diagram below. Establishing that the sequence is infinite is equivalent to establishing that s is not a real factor of d, and thereby that diagonal/side is an irrational number.

The actual relationship between diagonal and side has been found to be $d(d - s) = s^2$. The fibonnacci series is generated by starting with a non-regular pentagon of $s = d = 1$, and using the relationships, $d_2 = s_1 + d_1$, and $s_2 = d_1$. This relationship depends upon triangle ABC (in Figure 2) being isosceles.

Three students wondered whether the pentagon can be constructed from a circle. Although this suggestion was most certainly made in naivete, the question as to which polygons are

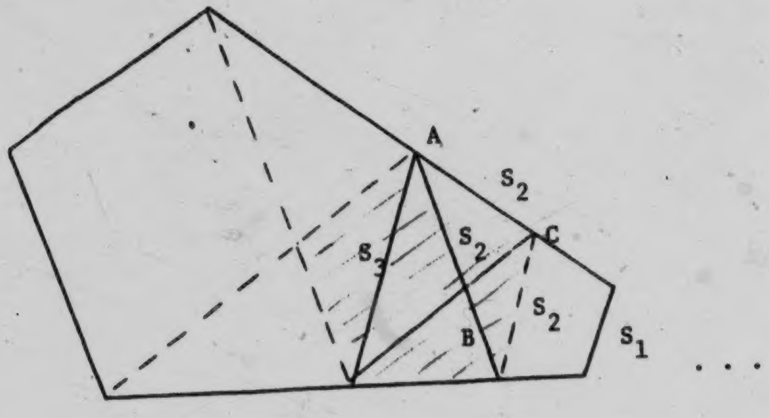


Figure 2

Relationships Between the Side and the
Diagonal of the Pentagon

constructable and which are not has been a historical problem. Gauss set up the following conjectures:

> Polygons were constructable if one of the following conditions held true:
>
> i) the number of the sides is a power of 2,
> ii) the number of the sides is a prime of the form $2^n + 1$ where n itself is a power of 2.
> iii) the number of sides is the product of primes satisfying the second condition.
>
> (Ley, 1967, p. 42)

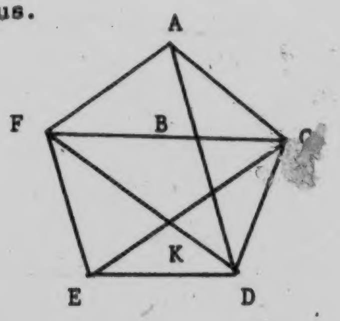
This type of detail for the pentagon is not important at the grade nine level, but the children can examine some polygons which are easily constructable and examine the reasons behind the constructions. The 2, 4, 8, . . . series is based on subsequent bisection of a diameter; the 3, 6, 12, 24, . . . series is based on the special characteristics of the hexagon, and subsequent bisections.

The decagon has as its base another construction determined by the golden section proportions, from which the bisection strategy can give rise to the polygons which have a multiple of five sides.

The quality of intuitive classification and logic can then be emphasized in geometry. "A geometric construction is supposed to be the result of pure thinking. . . .Since we know that the Greek geometers usually traced their construction in sand, the constructions certainly cannot have been accurate. But the thinking behind them was." (Ley, p. 33)

A fifth category of student response dealt with relationships in area and perimeter. Examples of these responses are numbers eight to fifteen listed on Table 25.

The pentagon opens up investigations in many areas of geometrical concept. Area, shape, angle size, the properties of polygons are some of the ideas which relate to concepts dealt with presently at the grade nine level. If desired, especially for the more advanced, formal proofs may be encouraged. Theorems such as "FBDE is a rhombus" or its equivalent "△DCK and △ACB are both isosceles triangles with two sides of s, and base, d - s" are important for many of the subsequent ideas and need justification--they are not obvious.



The above discussion has demonstrated that the pentagon meets many of the conditions necessary for a problem of good teaching potential, as specified by R. B. Davis. Davis discusses this characteristic of "many avenues of exploration" under the topic-extension approach, and emphasizes that a process approach to mathematics introduces the student to many concepts at the same time. "In the course of doing this, various other topics will appear which turn out to be more-or-less inextricably intertwined with the classroom work with the original topic". (Davis, 1967, p. 50) In his opinion, this not only increases the motivation and meaningfulness of the content, but enables the student to discover many of the concepts on his own terms, and gives him time to assimilate and to accommodate the new concepts. (Davis, 1964, p. 312)

The Triangles.

The Triangles question established itself as a more directed and less open-ended question than was The Pentagon. Table 26 lists some of the student responses in the first column, and then, in the second column, relates this conjecture to an area of study that may be generated by this conjecture.

Most of the meaningful ideas produced by the students dealt with measurement. This possibly was a result of the specific information included in the question and of the given hypothesis which related area and perimeter. In spite of the narrow range of response given to the problem, many of the same categories of study that were suggested for The Pentagon may be included for The Triangles.

TABLE 26

STUDENT RESPONSES TO THE TRIANGLES AS A SOURCE OF MATHEMATICAL STUDY

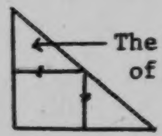
THE HYPOTHESES	AREAS OF INVESTIGATION
Using a right triangle, if you draw a square into the triangle as shown, then the area is decreased by ¼. The perimeter is decreased by ½. When does a line cutting one side of the triangle in ½ also cut the hypotenuse in half.  The area of this Δ is ¼ of the large triangle. If all the triangles joined together, what type of shape is formed? If another triangle, the same size was put beside the other triangle, in a certain way, they would become a triangle.	i) The study of proportion and similarity. ii) The use of deductive logic to establish such things as the base of the smaller triangle is ½ of the base of the larger triangle. iii) Mosaics. The historical account of the five Platonic Solids. Huntley, P. The Divine Proportion. iv) The development of area; the comparison with the parallelogram to establish the formula for the area for the triangle. v) The bisection of the triangles. Finding the various centers of the triangle. Under what conditions does the area divide in half?
An equilateral triangle when divided results in similar triangles. (I think the student meant congruent triangles.)	The circumcircle and the inscribed circle.
An equilateral triangle cut in half and put together differently form different shapes.	The study of right angled triangles. Trigonometric proportions.

TABLE 26 (Continued)

STUDENT RESPONSES TO THE TRIANGLES AS A SOURCE OF MATHEMATICAL STUDY

THE HYPOTHESES	AREAS OF INVESTIGATION
If one divides the triangles into two separate triangles, he would get ½ of the total area. What would happen if triangles straightened into right triangles? How would slant height vary as proportions of altitude and base very? When area is increased, the perimeter is increased. When perimeter is increased by one, the area is increased by two. Area is proportional to the altitude. The altitude is proportional to the area. Perimeter is affected by the altitude. When the perimeter is changed, the angle size changes.	The obvious concepts of area and perimeter, and their effects on each other. The multiplicative relationship of areas, the additive relationship of perimeter, the concept of altitude all seem to present a great deal of difficulty to students of this age. Even with the great deal of time spent on these concepts students are greatly surprised when they realize that the area of a triangle may be the same, yet the perimeter changes. One of the standard comments is--"If the perimeter of this triangle and that triangle is the same then it must enclose the same area." One student compared it to the following: "A short, fat container will contain the same amount of water as a tall skinny one so therefore the two different shapes must contain the same area." This calls for the very difficult and sophisticated task of examining the pertinent variables. The variable of altitude seems to be a troublesome one for many students.
	ii) A study of logical forms may also result. When the statement is true, the converse does not always have to be true.
	The study of similar triangles may also follow from an examination of shape and the conditions under which triangles increase in area and perimeter in the same

TABLE 26 (Continued)

STUDENT RESPONSES TO THE TRIANGLES AS A SOURCE OF MATHEMATICAL STUDY

THE HYPOTHESES	AREAS OF INVESTIGATION
	proportion.
If three sides of a triangle are equal to three sides of the second, both triangles have the same area.	1) The study of congruency and similarity.

(Table 26) The following are some of the general topics which may be investigated by students in a classroom.

i) The investigation of the existance of, and the conditions for tesselations formed by some combination of the triangles. This question may have been posed by the student conjecture: "If all the triangles are joined together what type of shape is formed?"

ii) The construction of polyhedra from the triangles.

iii) The investigation of the relationship between angle size and the size of the triangle. This may be area, the height-base ratio, perimeter or shape. The student conjecture "The altitude is determined by any pair of angles" falls into this category of inquiry.

iv) The investigation of the relationships between the circle and the triangles. A circle may be circumscribed on a triangle, or a circle may be inscribed. The latter opens up the investigation of the various centers of the triangle. A question may arise as to the type of triangle which may be inscribed in a circle and yet lie on the diameter. The conjecture "If one divides the triangle into two separate triangles he would get one-half of the total area" suggests one type of division for the triangle. (The idea of one-half would have to be made more explicit here, but this would also provide for inquiry and discussion.) There were no conjectures, given by the students, that were concerned with the circle-triangle relationships.

v) The investigation of the properties of special triangles. The conjectures "What would happen if the triangles were straightened out into right triangles?" and "How would the slant height vary as the proportions of altitude and base vary?" suggest one direction for study.

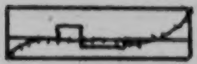
vi) The investigations of the area-perimeter relationships. These concepts have already been discussed in detail, and relate easily to the content and curriculum already present in the school system.

The first section in this chapter indicates the type of examination which may be done on the problem-situations for redefinition, sensitivity, and verification. This was not done at present because of the time and space limitations. The reader has already been provided with some information on the responses for the two verification problems and the second sensitivity problem during the categorizations established for the scoring procedures in Chapter V. A closer look at the responses given to The Board and Hole (Redefinition) and to The Square (Sensitivity I) is now being taken. This discussion just represents an abbreviated look at some of the ways students achieved responses to the two questions.

REDEFINITION

The Board and the Hole

The Board and the Hole offered an opportunity for some novel solutions which range along a continuum from a step by step analytical approach to a numerical, intuitive one as well as to a spontaneous inexplicable realization of the correct cut.

LN., (13-9, 124, 113) a very quiet student who performed at a low level on the conjecturing and verifying problems, was more at home with this problem and The Square. His ability in spatial relations seemed higher than his ability in logical relations--he quickly rejected the straight line cuts and thought immediately about the crooked line cut. He adjusted his initial thought  into the correct jagged cut, since the curved cut would present difficulties in fitting two halves together and suggested that his experience in woodwork made a response of this type reasonable.

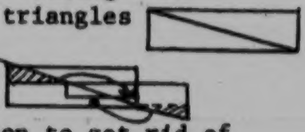
A solution on a second level was achieved by C. (discussed on page 236ff) who worked almost entirely from an analysis of the dimensions. She also achieved the solution in the first few minutes of the alloted time. Able to reject the obvious straight line cuts immediately, she foresaw the possibility of a solution. In her own words--"I had an idea of what the solution would be like." How or why was she able to realize this solution?

> I don't know, I just thought that way. The diagonal cut wouldn't work, the straight line across wouldn't work, the up and down cut wouldn't work. So I forgot about straight lines for a while and concentrated more not on the shape or on the way I was drawing my lines, but on the measurements, the way I had to draw them. The only problem I had was, for a while, I didn't know what to think about this 15 and 10, and then I didn't think of putting this right down the middle until I remembered they had to be equal, so I had to put this right down the middle and then I figured it out.

The measurements that C. refers to are the 3, 2 relationships between the sides of the board and the sides of the hole. She realized that the board had to be cut at the width of two inches from both sides. She continued this cut from both sides until she reached the middle and then realized that she had to make a vertical cut at this point. Note the step by step realization of the solution; she had the confidence to follow the intuitive idea of the final solution to its completion.

G. presented another analytical but spatial solution. His approach was characteristic of him. He conjectured (pages 235-236) by isolating the variables, then by varying the relationships of these variables, "I try to exhaust all the ideas on one variable." Here again G. realized that each piece had to be two inches on one end, one inch on the other and also that the dividing step was length five. Here is his explanation:

> I note that there must be two equal pieces, so I tried a simple idea and from that idea I might develop other ideas which might help me in finding the solution. So I divided this into triangles and I just slipped it like this so that you get two here and two here so that you have to get rid of those, so I thought of a solution to get rid of those and that they both are equal, so one would fit here, and the other fit neatly here, then I tried it out and it worked. I then tried to get another idea.



GN. (14-8, 94, 98) exemplified the other extreme in response to The Board and Hole problem. GN., operating at a low level of abstraction, was very dependent upon the visual material presented and tended to ignore numerical analysis. Her low level of maturity was indicated by her response to the question of whether the shape of the final answer was familiar to her. This was an unusual shape for her; usually one worked with triangles and rectangles, but this was a different shape. Shapes like these were used for building castles. (She had seen a shape like this in the structure of an old castle on TV).

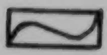
Below are the various stumbling blocks that hampered her as she proceeded to find the solution:[7]

i) She traced the diagrams onto the graph paper and counted the squares within rather than use the numerical dimensions given. The visual facts were more important to her. (Soviet Studies, p. 115)[8]

ii) She then proceeded to cut the board in three, then the third piece in half. She was unable to see the inappropriateness of this until asked to reread the instructions.

iii) She suggested that the idea of a zig-zag cut had passed through her mind, but it had seemed incorrect since the final requirement was to be a rectangle.

iv) Her approach was to try something new each time. She was not able to evaluate what she had done in terms of improving her next attempt. Therefore her zig-zag cuts (after the idea was suggested) tended to be random rather than directed responses.

GN.'s difficulties are described because these stumbling blocks were common to others. IJ. (13-8, 130, 143), an individual who solved the problem quickly by taking the  cut, and subsequently adjusting this idea by using the factors of 2, 3 thereby producing the correct solution, foresaw the following two stumbling blocks which he thought might prevent others from achieving the correct solution:

---

[7] GN. Although GN.'s lack of mathematical knowledge and skill hampered her progress in the experiment, she nevertheless produced a very intuitive and elegant response to Verifying I parts a, b. Her drawings of the square and the triangle, and their increased counterparts as well as her discussion, showed clearly that she "squared" proportion, and had some idea of the necessity of similarity.

[8] The experiments showed that although the object under analysis (the diagram) was the same for all subjects, they saw different things and different numbers of things, depending upon the level of their processes of analysis and synthesis.

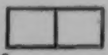
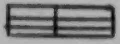
i) People usually don't think that equality and accuracy can be achieved with a curved line. You can't cut exactly in half that way.
ii) When looking at the board a jagged cut doesn't look as if it would fit the hole. Why? Both have sides that are straight and regular. Thus it doesn't seem as if a jagged cut would fit.

Some individuals who realized that a crooked line cut was necessary looked for something more complex. They were reminded of the cross-word puzzles and looked for interlocking pieces. GM. was one who overcame this obstacle.

> GM. I knew that if you put in too many cuts or turns you would have to synchronize them to fit together. So the more turns the more difficult it is to get them to fit together.
>
> Q: What experience helped you to realize this idea?
>
> A: Not really an experience but even the work on fitting the polygons (Pentagons as a mosaic). You realized that if you have too many sides then you have a harder time fitting them together.

This individual was able to encorporate previous experience to a problem not usually associated with it. Kabanova-Meller (1950) has suggested that previous experience may either help or hinder an individual in solving problems depending upon his level of abstraction of the original experience.

Three individuals who misread the problem and did not realize the significance of the instruction, "two equal pieces," illustrate the effect of previous directed experience on ability to achieve solution. These boys cut the board into more than two pieces and fit the pieces onto the hole. When their attention was drawn to the instruction of "two equal pieces" they were able to realize the correct cut immediately.

One special cut into four pieces (two long, two short) provided a good intermediate cut. A variation of this cut, accidently made by S. (14-7, 148, 126) caused her to see the solution. She used the graph paper, and automatically made the cut  . She rejected this but tried the horizontal cuts on the same diagram, knowing they were incorrect but essentially drawing them twice, as shown.  She then saw the correct cut just by absently looking at the diagram. S. had a very good spatial rearranging ability. She was one of the three individuals who were able to see the parallelogram in the sixth shape of Redefinition B-Areas.

It is notable that S. produced only straight line cuts in response to The Square. S. scored at the average level on the novelty aspect of conjecturing although she produced an average number of responses for The Pentagon and an above average number of responses for The Triangles. S.'s forte was in directing her talents toward a specific question and she very quickly solved such problems. Her conjectures were of the observational type, and even for the triangle problem which included numerical data, she did not do any quantitative work. Specifically her conjectures were:

i) None of the triangles are right triangles.
ii) The base is the same and when the figures are put together, they would make a shape with many points,

but the same base.

iii) The hypotenuse of a triangle is larger with more gradual slope and largeness of the opposite angle.

iv) When two triangles are put together, the resulting figure is a parallelogram.

Several high IQ, high achievement individuals were not able to direct their ideas to a solution. Their reaction to the words, two equal pieces, was "I didn't realize we could cut it this way." R. (15-0, 134, 118) was one of these. Until confronted with the definite statement that there was a solution, he steadily maintained that the cut could only be made in depth. Most of the individuals who were able to solve this problem and certainly those who provided most of the responses for The Square looked upon the concept of one half from the point of view of areas rather than from the ideal of symmetrical pieces. Some incorporated the idea of symmetry, others did not, but the dominating feature of "one-half" as area often appeared to remove a set toward the straight line cut.

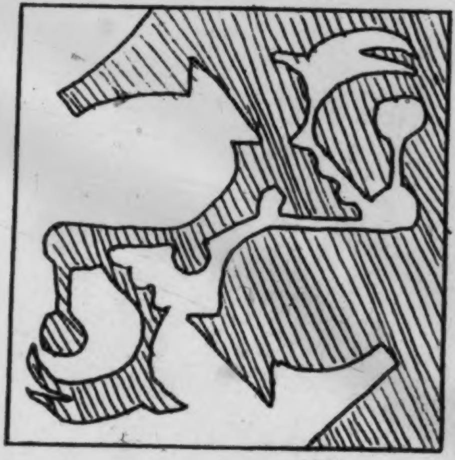
SENSITIVITY

The Square

The following section presents some of the types of responses that were produced in response to the instructions "Cut the square in half. What are the shapes of the resulting halves?" The types of responses have been listed in an approximate hierarchy, from most creative to least creative. To some extent, the students were aware of the variety of responses that could exist because all of them had been exposed to The Board and the Hole problem reported in the section before.

i) One individual produced four or five several distinct categories of shapes which could result from cutting the square in half. He not only considered the zig-zag cuts but cuts from inside the square and various curve cuts. This individual responded to the self-imposed instruction: "How many different shapes can result and how can they be classified?"

ii) A second individual produced several types of cuts, all different, then proceeded to draw a very elaborate design which cut the square in half. The design was symmetrical as well as covering half the area in each piece.



iii) A third group of individuals produced several cuts, using straight lines, curved lines and zig-zag lines, but stated a generalization to describe how to cut a square in half: "Do exactly to one side that you would do on the other." These persons realized that there were an infinite number of cuts, however were not stimulated to classify or to describe the classes which may have existed.

iv) A fourth group of individuals produced several shapes as in iii), realized "There are an infinite number of shapes.", but did not state any generalization or show any intent to investigate a classification scheme.

v) A fifth group of individuals produced a simple classification similar to that in number i), but dealing only with simple shapes, such as triangles, rectangles, and squares. The student exemplifying this category was inspired by the instruction "How many different shapes can you find?", and started to develop halves which matched the polygons that he knew.

vi) A sixth group of individuals divided the square an infinite number of times by progressively cutting each half in half. This can be done by initially dividing the square into a rectangle or a triangle.

vii) A group of individuals produced straight line and zig-zag cuts. These students remembered The Board and Hole problem, however had solved the former problem, indicating that they had not been inhibited by the set that a half had to be a straight line cut.

viii) A group of individuals as in number vii), but these students had not solved The Board and Hole problem, and thus were influenced by the learning that had occurred at the time.

ix) A group of individuals who produced only straight line cuts. These students had solved The Board and Hole problem but were unable to apply this knowledge to the open-ended question. These individuals were often very intelligent, very high achievers or both.

x) A group of individuals who produced an infinite number of straight line cuts.

xi) A group of individuals who produced a finite number of straight line cuts. Some of these students were aware that there may be more cuts, although a finite number, but many of them would state that there were not any more ways of cutting the square in half.

The responses to redefinition or sensitivity have not been classified or investigated fully, however, the following comments may summarize some of the researcher's observations.

An individual's type of approach may remain constant from one situation to another. G.'s analytical consideration of the variables, and the step by step application of logic appeared in both the conjecturing problem and the redefinition problem. G. also defined the first category for The Square. Here again he classified at the abstract level. In Vygotsky's sense he is probably at the full concept level. C. also produced responses in a characteristic way, although she showed the ability to be very effective in the directed thinking situation of The Board and the Hole. She indicated both the whimsical and the commitment attitude (see her statement on page 237) in her response to The Square (Category ii).

The effect of previous experience on the solution of a problem varies greatly. Some individuals did not relate even very close experience, like The Board and Hole and The Square. Others related questions like The Pentagon and the redefinition situations. Insight about the individuals who related experience and those who didn't may result from a more detailed examination of the three remaining processes.

The discussion in the above chapter has tried to provide some description about the student response to the constructed questions. The main emphasis has been on the conjecturing process. The oral interpretations given by the students provided clarification of the written responses, and this chapter has tried to provide the reader with some of the insights available to the researcher.

This discussion has also tried to establish some of the ways in which students conjecture. It has been shown that in many ways, the intuitive procedures used by the student in the sample can be classified under procedures suggested by mathematicians. The various levels summarizing the responses emphasize the importance of the student's confidence and facility with basic ideas before relationships occur. The stage of stating only observed facts seemed to be a necessary step previous to the step of producing associations.

CHAPTER VIII

SUMMARY AND IMPLICATIONS

The present study is concerned with the procedures by which students solve problems in geometry. In order to determine the extent to which junior high school students can react creatively to mathematical situations, a model for creative thinking was developed with reference to the literature on creativity. Four processes for creative problem solving were defined, and two problems that theoretically reflected each process were constructed. The responses to the problems were scored and these scores correlated. The correlations were examined by pairs and by the procedure of image analysis in order to determine whether the hypothesized four processes emerged on the basis of the responses given by the student sample.

The image analysis yielded six factors. One factor was entirely identified by the Torrance tests. A second factor corresponded to the conjecturing process, and a third to the verifying process. A fourth factor, not as clearly delineated by the constructed situations, was defined by the redefinition and ability scores, and was thus interpreted as the redefinition process. The two sensitivity problems separated into the two remaining orthogonal factors. Because of the complexity and the difficulty level of Area-Increase, it was decided that The Square was a better representation of the process of sensitivity.

Although the interpretation of the four factors seems consistent with the results, and will be accepted for the purpose of this study, it is possible that the four processes are composed of sub-units. These sub-units may involve the five factors, other than the Torrance tests, which separated under the image analysis. Sufficient information to determine or identify these sub-units was lacking, but some of them may be identified in the eight summary points included in the final paragraph of this section.

Several difficulties were apparent in the problem-situations. The second redefinition problem is not clear in its purpose. Parts iv) and vi) in Redefinition II, Angles, may be solved in several ways, and it is not clear that one specific method should be considered superior to another. (pages 139-140) The particular method of solution used was not clear from only the written responses. Since scores were awarded on the basis of method used, the responses given during the oral interview were instrumental in determining the score. It may be argued that only part v), the item that could be solved in only one way to get the correct answer, and the one way that required the breaking of the set, should be awarded a score. The same comment applies to Redefinition IIB, Areas. In addition, the crucial part in Redefinition II, Areas, that is, part vi), seemed too difficult for the sample of students. Only two individuals were able to solve the problem without help. The type of scoring used may have affected the correlations between redefinition and the other scores and thus the interpretation of redefinition. The criticism may be answered by further research. Further questions about the constructed problem situations and their administration are raised in the last section, Implications for Research.

To what extent are the characteristics and guidelines established for each process valid? It may be that this question is irrelevant, and it may be important to see if the four defined processes appear as separate entities when responses to random problem-situations are examined. On the other hand from the measurement point of view it is convenient to be able to focus on one process as much as possible.

Both The Pentagon and The Triangles were found to meet the requirements of conjecturing. However, the content used in Conjecturing II resulted in the use of other processes by the students. The data presented to students in Conjecturing II was much more familiar to the students than that presented in Conjecturing I. The given conjecture re-emphasized the familiar interp[illegible]tion. The pentagon in Conjecturing I was a new object and most of the responses emphasized the observable generalizations rather than the numerical data. In time, as the students learn more about the pentagon, the aspects of verification and redefinition may appear in the responses. In summary, the conjecturing process may be contaminated less by other processes, if care is used in selecting a problem or situation which does not have many connotations to the student. However such responses may be more superficial and less quantitative than if the situation is a familiar one.

Sensitivity I seems to be a fair representation of the established guidelines for this process. It may be, however that in the guidelines the presence of a "set" should be de-emphasized. A situation for which many responses are possible should perhaps be presented and students should be made aware that the possibility for many responses is present. Sensitivity should be an awareness of some unusual response among the many, rather than just the realization of divergence in what seems to be a convergent problem. This idea is simple to conceive, but difficult to interpret in the construction of problems. First there is the difficulty of how to instruct the student without telling him that there are unusual possibilities in the situation. Second there is the difficulty of distinguishing and separating this process from that of conjecturing. More experimentation with problems using the instruction: "What are the shortcomings in this data?", or "Suggest the possibilities for problems which are present in this data." are necessary before alternate and clear guidelines can be presented. It also may be that at the junior high school level conjecturing and sensitivity are difficult to separate under this type of instruction because of the students' "question" approach to the former. (The students proceed in making conjectures by asking "I wonder if." or "What happens?")

The guidelines for redefinition require that the students overcome or resist the inclination to use an inefficient or an incorrect method to a problem which seems similar to previous and successful experience with this established method. Although both redefinition questions meet this requirement in spirit, difficulties resulted in the establishing of the set and in the scoring of the "set-breaking" items. The first three items in Redefinition IIA, and Redefinition IIB were not strong, they did not firmly establish one definite approach for solution. Second, the decision to score the questions which could have been solved in two ways may not have been a valid one because of the difficulty in establishing which approach is really the most appropriate. From this point of view it may be more valid, especially if written responses are to be scored, to construct a redefinition situation with six items, five of which establish a certain set and one which can only be solved by an alternate method. This sixth item should be at an appropriate level of difficulty for the group in question. Students found item vi), in Redefinition IIB, Areas, difficult to solve even after they were given specific hints.

For verifying, the instruction "Prove each statement as best as you can" seemed appropriate and, as desired, elicited responses from testing to responses of abstract argument. Two difficulties appeared in the verifying problems. First, specific numerical data should not be given. A problem with this type of information (Verifying IIB) seems to correlate with the redefinition questions. Second, a problem that does not allow the student to test with specific numbers may be too difficult in that it is not interpreted as a problem by the student but only as a statement of "truth." The responses for Verifying IIA, Parallel Lines, were often just restatements of the original problem, at times accompanied by some measurement of angles or lines in the physical diagram. This restatement seemed obvious to the student, and so for him no problem existed. Verifying IIA, Parallel Lines, is still appropriate as a verifying situation and the responses discussed during the oral interview were interesting and informative. However, statistically it may be invalid because of the many individuals who were awarded a score of zero on this problem. (30/42 received a score of zero, five more received a score of one)

The particular problems chosen to represent the four processes werefound to interrelate and appear to depend upon cognitive approaches which are correlated. The requirements between problems which seemed similar and the specific correlations have been discussed in detail in Chapter VI and Chapter VII. In summary, some of these intercorrelations suggest the following:

i) The ability to isolate important variables in a problem-situation seems to have been an important requirement in verifying and in conjecturing.

ii) The ability to foresee the dependent and independent variables seems to influence performance on the conjecturing questions, the verifying questions, and Sensitivity II.

iii) The ability to foresee a given result, and to see the data from several perspectives seems to result in effective responses for the sensitivity situations, for the redefinition situations and for the verifying situations.

iv) The ability to follow a logical argument in a step by step fashion seems to result in some of the more efficient responses for Redefinition I, for Verifying I, and for the conjecturing situations.

v) The ability to be aware of possibilities beyond those suggested by the data given seems to be necessary for Conjecturing I, and for the sensitivity problems.

vi) The ability to break from a set established by previous experience or by data in the problem seems to determine the range of responses given to the sensitivity questions, the redefinition questions and to Conjecturing II.

vii) The ability to formulate relationships in response to an open-ended situation seems to be distinct from the ability to find the most appropriate response specified by a definite requirement. These abilities were described as divergent and convergent, respectively.

viii) The ability to generate relationships, to formulate conjectures, seems to be recognizable to the students. Some of them were unable to do more than list observable facts, but were aware of the inappropriateness of their responses.

The data in this study does not repudiate the model. There is some difficulty in interpreting the isolation of Sensitivity II from the other factors and, as indicated above, the processes may be a conglomerate of abilities. However, acceptance of the four processes aided in the interpretation of the statistical results. The responses given to the problem-situations indicate that the grade nine students in the sample were capable i) developing conjectures about given data, ii) verifying and testing stated conclusions, iii) manipulating specific variables and in breaking with traditional understanding in order to achieve a specific result (redefinition), and iv) manipulating specific variables in order to demonstrate unusual understanding of a seemingly obvious situation and as a result, producing new and novel possibilities (sensitivity).

IMPLICATIONS FOR THE CLASSROOM

A basic premise underlying this study is that if creativity is desirable as a product of education, then it should be emphasized in teaching. The hypothesis is that creative effort is possible from everyone, although at different levels, in response to varying situations. It also seems that the hypothesized processes are valid in a discussion of creative endeavor in mathematics. The experimenter would like to contend that the teacher of mathematics should incorporate these processes into the classroom work.

The constructed situations are most valuable because of their potential as teaching situations. Some indication of this potential, with regard to mathematical content and its interrelationships has been made in Chapter VII (pages 259-271). Table 25 and Table 26 list descriptions of topics which may center around a student response to the conjecturing problem-situations. If a student conjecture is used to initiate or supplement a classroom discussion, student ideas may establish questions and investigations along the areas suggested. Within the limited context of the given study, two students were stimulated to extend the ideas discussed in the experiment by doing some individual investigation at home. This type of approach may be rewarding to the teacher in that it provides him with the opportunity to also be a student of mathematics and to react to the new ideas that are often posed in an open-ended situation.

One objective of the study was to experiment with an evaluation instrument. The importance of establishing procedures for evaluating the four processes is a practical one. Most children learn that on which they are evaluated and have learned to place importance on the type of thinking that is rewarded by our grading system. The scoring procedures used in the study, although time consuming, indicate some possibility for emphasizing and evaluating student response to divergent situations. The variety score (shown to correlate with the novelty score) may be of practical use to the classroom teacher. After some initial experience with the isolation of concept sets, the variety score may be approximated. This approximate score would provide a fairly simple and effective assessment of a student's ability to respond to situations such as those of conjecturing and sensitivity. Divergent process, as in conjecturing or in sensitivity, may be an important means by which knowledge is assimilated as well as generated. Learning by conjecturing may be much more encompassing than learning by repetition; it may mean relating of ideas and skills into some meaningful total. The importance of divergent process is yet a matter of opinion. Further research must uncover its potential as well as its application.

There is a practical difficulty both in the teaching and evaluation of such situations. Seven classes with thirty or more students per classroom do not provide the teacher with much opportunity to examine and to encourage individual thinking processes in any detail. Nevertheless, this type of teaching and study into the students thinking allows the teacher the opportunity to be a student himself. The knowledge from this type of examination may also result in more effective teaching as well as learning.

IMPLICATIONS FOR FURTHER RESEARCH

Some further research is suggested by the results and by the difficulties and the limitations present in this study. These are presented as a series of questions in the sections which follow.

The first set of questions consider variation in the administration or construction of the problem-situations.

i) What would be the effect on the type and the number of responses given to each conjecturing question if the original sample conjecture was changed? Would the emphasis on area and perimeter in the responses to The Triangles be lessened if the sample hypothesis dealt with another relationship, say mosaics? Would there be an increase in the number of quantitative responses for the pentagon if the sample conjecture for this situation dealt with area and perimeter?

ii) What would be the effect of a practice session on the number of responses given to the conjecturing or to the sensitivity problems?

iii) What would be the effect of substituting another polygon for the pentagon? As a learning situation, the pentagon has great potential; as a testing situation, it may be too much the unknown quantity. The difficulty lies in choosing a more appropriate polygon. Some individuals may have been exposed to the hexagon by a given teacher within a specific classroom, others may have discussed it at only a superficial level. When this happens it is difficult to distinguish between responses creative to the individual and those discussed by the teacher. The choice might be to use a polygon that is very familiar to everyone, say a square or a triangle, and then to ask the students to generate conjectures. Would the results be comparable to those achieved with the pentagon? Would the conjecturing problem about the square identify as distinct a factor under image analysis as had the pentagon problem?

iv) What is the reliability of the test scores? If the given problems are to be used as an evaluation instrument, the scoring procedure must be examined carefully, particularly at the points where categorization takes place. An appropriateness and classification scales were used in the present study. This classification should be compared with categorizations rendered by expert judges and standardized scoring schemes established. Such a reliability check could entail a study in itself because the categorizations and the analysis of the answers as well as of the clarifications given orally are time consuming.

v) What is the effect of order of presentation of the eight problems on the number and quality of responses? An example of the small studies possible from the data gathered is included in Appendix A. This report deals with the question "Did the students working on The Pentagon fourth in the problem sequence provide more novel responses than did those who did the problem first?" The results were negative.

Further research is necessary to extend the context in which individuals produce novel responses. Extensions on this study may occur in the following ways:

i) Can a number of distinct problem solving styles be isolated? The oral discussions with certain individuals, such as G., C., CL., and GM. suggest that, although these individuals recognized the difference between open-ended and closed problems, they tended to approach any situation in a characteristic style: G. summarized and analysed the variables, C. continually asked "what if?" and tended to "follow her nose"; CL. looked for differences between things he knew and things he didn't know; GM. tended to use a very intuitive idea of basic ideas to develop new numerical relationships. Attitude and problem style may influence quality or quantity of response. Knowledge about specific problem-solving approaches may influence presentation of material and the attention of teachers to specific learning factors.

ii) How do the students' responses to the constructed problem situations in this study compare with responses to the problems constructed by Taylor-Pearce? The Taylor-Pearce problems depend upon mathematical ideas at the grade eleven or twelve level. The concepts in the questions constructed for the study, on the other hand, should be very familiar to grade eleven students, enabling them to be as creative as possible. (Redefinition I, Redefinition II, Angles, and Verifying I may be too simple and therefore invalid at this level, but this is a supposition which needs to be investigated.) Would the same students who score high on the constructed questions also score high on Taylor-Pearce's problems? Do the two sets of problems measure the same domain or would further factors emerge?

iii) What is the effect of training on conjecturing, sensitivity, redefinition, and verifying? Luchins has already suggested that one of the most valuable ways of breaking the tendency for the formation of specific sets is by exposing individuals to such experiences and then discussing the formations of their sets with them. It has been the researcher's experience that this is effective with junior high school children. By exposing them to such situations and then exposing to them their sets, a set towards being flexible and an attitude which tends to establish an alertness for the unusual tends to be fostered in the student.

Similarly, it may be supposed that the variety and the quality of hypothesizing will increase if some discussion of student-made responses is carried on in the classroom. Discussion of how such a hypothesis was made and what prompted it should serve to make the students more capable in generating other responses. This is the old story of teaching specifically the ideas desired for retention and for transfer.

The present study has been exploratory in nature. It has enhanced the researcher's understanding of the type and level of approaches used by students in solving problems. It has also raised many questions on concept formation. Skemp and Lovell have tried to explain the development of concepts and schema for mathematical learning by incorporating the theories on concept formation of people like Vygotsky with those developed by psychologists concerned with constructs such as receptors and intervening variables. The Soviet Studies in the Psychology of Learning and Teaching Mathematics Series (1970) records the extensive work of many experimenters who used a procedure similar to the one used in this study to seek out pertinent factors in mathematical problem-solving. For example, in Zykova's article "Operating with Concepts when Solving Problems", problems were presented to individuals of varying abilities and their answers described in detail in an attempt to distinguish between the type of solution provided by the high ability individual as compared to that provided by the low ability individual. There is much data on the tapes made during the present experiment that could be examined in light of these studies and ideas.

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APPENDICES

APPENDIX A

THE EFFECT ON PERFORMANCE OF THE ORDER OF PRESENTATION OF THE TWO CONJECTURING SITUATIONS

THE EFFECT OF THE ORDER OF PRESENTATION OF TWO PROBLEMS ON THE FLUENCY, VARIETY AND NOVELTY SCORES FOR THE RESPONSES

The purpose of the investigation presented here was to investigate whether students who solved Conjecturing I, The Pentagon, first, then solved Conjecturing Two, The Triangles, performed differently from those who solved Conjecturing II, first, then solved Conjecturing I. These problems, two of the eight problems presented to forty-two grade nine students from the Edmonton Public School System, were presented in two orders in order to overcome practical difficulties in the accommodation of the students during the experiment discussed in the body of the main text. Half of the sample worked the problems in the order: Conjecturing I, Sensitivity I, Redefinition I, Verifying I, Conjecturing II, Sensitivity II, Redefinition II, Verification II. The second half worked the problem in the order: Conjecturing II, Sensitivity II, Redefinition II, Verifying II, Conjecturing I, Sensitivity I, Redefinition I, Verification I. The following two hypotheses were formulated in an attempt to investigate the effect of order in which problems were solved on performance in each problem.

The children had chosen the time at which they were to come for the experiment, so whether student A solved The Pentagon first or The Triangles first was a matter of accident. Thus, it was first necessary to determine whether the group who solved The Pentagon

(Triangles) first, differed in ability from those who solved The Pentagon (Triangles) second. The SCAT scores were chosen for comparison because this test is a reflection of ability and is also a good predictor of achievement in the school situation. (Table 27) The F test for homogeneity of variance indicated that the variances of the two samples were representative of the same population. ($p < 0.05$) The t values comparing the means for the two groups were very close to zero, supporting the assumption that the two samples were comparable in ability, and that a comparison of the two groups with respect to their performance on the two orders of presentation of the conjecturing situations was plausible.

The means and the variances for the fluency, the variety and the novelty scores for the situations were calculated, the F test and the t test again applied. These scores, as well as the values for these statistics, are shown on Table 28. The F values were not significant for the pentagons situation and the t test failed to reject the hypothesis that the means for the two samples were representative of the same population mean.

The student scores on Conjecturing II, the statistical values for F, homogeneity of variance, and for t, the Students' Statistic for the Difference of Means, are given on Table 29. Since the assumption of homogeneity of variance was not justified for the novelty and the variety scores ($p < 0.05$), the Welch correction for the t test was used.

The difference between the two sample means was not significant at the $p < 0.05$ level for any of the three measures and in fact was very close to zero for the novelty score. The conclusion

TABLE 27

THE SCAT SCORES FOR THE SAMPLE OF STUDENTS DIVIDED INTO TWO GROUPS FOR WHICH ORDER OF PRESENTATION OF PROBLEMS DIFFERED

A COMPARISON OF THE SCAT SCORES OBTAINED BY GROUP A AND GROUP B

#	A-I SCAT N.V.	#	B-I SCAT N.V.	#	A-II SCAT VERBAL	#	B-II SCAT VERBAL
1	9	22	8	1	9	22	8
2	7	23	9	2	9	23	7
3	8	24	8	3	8	24	7
4	9	25	8	4	7	25	8
5	9	26	9	5	7	26	9
6	7	27	9	6	8	27	8
7	5	28	8	7	7	28	6
8	7	29	4	8	8	29	7
9	9	30	5	9	7	30	6
10	8	31	6	10	6	31	6
11	7	32	6	11	8	32	5
12	6	33	8	12	4	33	8
13	9	34	4	13	5	34	4
14	6	35	5	14	6	35	6
15	6	36	7	15	5	36	7
16	8	37	5	16	5	37	4
17	4	38	6	17	7	38	9
18	5	39	8	18	5	39	8
19	7	40	6	19	6	40	6
20	6	41	5	20	9	41	7
21	5	42	6	21	5	42	6
ΣX	140		140		133		142
$\frac{\Sigma X}{N}$	7.0		6.67		6.65		6.76
ΣX^2	1028		988		929		1000
sX^2	2.40		2.61		2.23		1.90
N	20		21		20		21

STATISTICS: $F = 1.09$ (Calculated)
$F_{.95}(19, 20) = 2.07$
$t = .685$ (Calculated)
$t_{91.5} = 2.02$
$t_{95} = 1.68$

STATISTICS: $F = 1.16$ (Calculated)
$t = .250$ (Calculated)

(A--Pentagon-Triangles; B--Triangles-Pentagon)

TABLE 28

A COMPARISON OF THE SCORES OBTAINED BY GROUP A AND GROUP B FOR THE PENTAGON

GROUP A	HYPOTHESIZING ONE			GROUP B	HYPOTHESIZING ONE		
No.	Fluency	Variety	Novelty	No.	Fluency	Variety	Novelty
1	3	2	1.5	21	4	4	4
2	3	3	2	22	3	2	4
3	2	2	1.5	23	3	2	0
4	3	3	0.5	24	2	2	0
5	2	1	2	25	4	4	3
6	5	4	3	26	3	2	1.5
7	4	3	1.5	27	2	1	0
8	1	1	0.5	28	1	1	0
9	0	0	0	29	4	3	1.5
10	3	2	1.5	30	5	3	4.0
11	4	2	0.5	31	1	1	1
12	2	1	1	32	2	2	1
13	5	2	0.5	33	4	2	0.5
14	4	3	2	34	2	2	1.5
15	4	2	1.5	35	4	3	1.5
16	3	2	3	36	6	3	0.5
17	6	3	1.5	37	4	3	3
18	4	2	1	38	3	2	1.5
19	4	2	1	39	4	2	0.5
20	3	3	4	40	3	3	1.5
				41	3	2	0.5
TOTAL	65	43	30		67	49	31
MEAN	3.25	2.15	1.50		3.19	2.33	1.48
Σx^2	249	109	63.5		245	129	82.5
s^2	1.9	.83	.92		1.48	.70	1.75

*Group A received the pentagon problem first.

STATISTICS: F test for homogeneity of variance and student t test

$H_A \quad U_a - U_b = 0$ $\qquad t_{fluency} = .159$ $\qquad$ df = 20+21-2 = 39

$F_{Fluency} = 1.65$ $\qquad t_{variety} = -.23$

$F_{Variety} = 1.185$ $\qquad t_{novelty} = .049$

$F_{Novelty} = 1.89$ $\qquad t_{99.5} = 2.70$

$F_{.95}(19, 20) = 2.07$ $\qquad t_{.05} = 1.68$

TABLE 29

A COMPARISON OF THE SCORES OBTAINED BY GROUP A AND GROUP B FOR THE TRIANGLES

GROUP A	HYPOTHESIZING TWO			GROUP B	HYPOTHESIZING TWO		
No.	Fluency	Variety	Novelty	No.	Fluency	Variety	Novelty
1	5	3	1.5	21	5	5	4
2	3	2	2.5	22	6	5	4
3	4	1	2	23	4	1	2.5
4	3	2	0	24	3	1	0
5	3	1	2	25	7	3	3.5
6	3	1	0	26	3	2	0
7	3	2	2	27	3	2	0
8	4	2	2	28	2	2	0
9	2	1	1.5	29	4	2	0
10	4	1	0	30	5	2	0
11	0	0	0	31	2	1	0
12	4	1	0	32	2	1	0
13	1	1	0	33	2	1	0
14	5	2	1.5	34	3	1	0
15	4	3	2.5	35	3	2	0
16	2	2	2	36	4	2	0
17	1	1	0	37	6	3	4
18	1	1	0	38	4	4	2
19	2	2	2	39	4	2	0
20	2	2	4	40	2	1	1
				41	1	1	2
TOTAL	56	31	27.5		75	44	23
MEAN	2.80	1.55	1.48		3.57	2.10	1.10
ΣX^2	204	59	53.25		327	124	75.50
s^2	2.36	.45	.77		2.81	1.51	2.40

*Group B received the triangles problem first.

STATISTICS: F test for homogeneity of variance and student t statistic

H_B $U_a - U_b = 0$

$F_{Fluency} = 1.2$

$F_{Variety} = 3.35$

$F_{Novelty} = 3.12$

$F_{.95}(19, 20) = 2.07$

$t_{fluency} = 1.53$

$t_{variety} = 1.80$

$t_{novelty} = .09$

degrees of freedom

Corrected (Welch) = 39; $t_{97.5}(39) = 2.02$

Corrected (Welch) = 32; $t_{97.5}(32) = 2.04$

Corrected (Welch) = 31; $t_{97.5}(31) = 2.04$

then seems to be that it did not make any difference in an individual's score whether he met the pentagon situation first or the triangle situation first.

APPENDIX B

THE FIVE-FACTOR SOLUTION FOR THE IMAGE ANALYSIS PROCEDURE CARRIED OUT ON THE TWENTY-SIX TEST SCORES FROM THE TORRANCE TESTS, THE SCAT TESTS, AND THE CONSTRUCTED TESTS

An image analysis, calling for five factors, was performed on the twenty-six scores (The Torrance tests, the SCAT tests, and the constructed tests). The loadings on the five factors is presented in Table 30.

The five-factor solution is consistent with the discussion for the six-factor solution used in the main text. The Torrance tests loaded on a distinct and separate factor, Factor IV', emphasizing the lack of correlation among these scores and those from the constructed tests. Similarly the loading on Factors II', III', V' were almost identical to the loadings on Factors V, III, VI from the six-factor analysis. Factor II' was characterized by Area Increase, Factor III' by the conjectured problems and Factor V' by The Square. The tests that had loaded on Factors I and II on the six-factor solution merged into Factor I'. These were the situations measuring the verifying and the redefinition processes, indicating that the four convergent problems seemed to share more traits in common than any other grouping of the eight constructed tests. The SCAT tests also loaded on this factor (0.36, 0.52).

The strong verbal influence in Sensitivity II, Area-Increase, is again emphasized by the 0.41 loading of the SCAT verbal on Factor II'. Factor V', mainly described by The Square, Sensitivity I, also shows loadings from the SCAT non-verbal (0.39), Redefinition (0.30), Verifying II (0.50) and Conjecturing II (0.40). These loadings seem to be consistent with the discussion in the main text. These indicate the presence of a multiple step requirement in the problem, that of realizing the presence of unusual ways of cutting the square in half and that of the redefinitive ability in achieving the break from straight line cuts.

TABLE 30

FIVE IMAGE FACTORS EXTRACTED FROM TWENTY-SIX SCORES REPRESENTING CREATIVITY, ABILITY AND THE CONSTRUCTED PROBLEMS

VARIABLE	I'	II'	III'	IV'	V'	COMMUNALITY
SCAT Verbal	.36	.41	.36	-	-	.45
SCAT Nonverbal	.52	.34	-	-	.39	.56
Torrance Fluency	-	-	-	.97	-	.94
Torrance Flexibility	-	-	-	.91	-	.85
Torrance Originality	-	-	-	.97	-	.95
Conjecturing IF	-	-	.65	-	-	.56
Conjecturing IV	-	-	.78	-	-	.65
Conjecturing IN	-	-	.81	-	-	.71
Conjecturing IIF	.37	-	.59	-	-	.54
Conjecturing IIV	.55	-	.60	-	-	.73
Conjecturing IIN	.30	-	.46	-	.44	.56
Sensitivity IF	-	-	-	-	.69	.49
Sensitivity IV	-	-	-	-	.88	.84
Sensitivity IN	-	-	-	-	.71	.83
Sensitivity IIF	-	.87	-	-	-	.88
Sensitivity IIV	-	.95	-	-	-	.93
Sensitivity IIN	-	.82	-	-	-	.68
Redefinition I	.58	-	-	-	-	.40
Redefinition IIA	.60	-	-	-	.37	.54
Redefinition IIB	-	-	-	-	.31	.21
Verifying Ia	.80	-	-	-	-	.76
Verifying Ib	.67	-	-	-	-	.45
Verifying Ic	.80	-	-	-	-	.69
Verifying Id	.81	-	-	-	-	.77
Verifying IIA	.55	-	-	-	-	.35
Verifying IIB	-	.39	-	-	.48	.47
TOTAL COMMUNALITY	4.69	3.05	2.99	2.95	2.88	16.56
% COMMUNALITY	26.2	18.4	18.2	17.8	17.4	

*Only loadings above .30 reported.

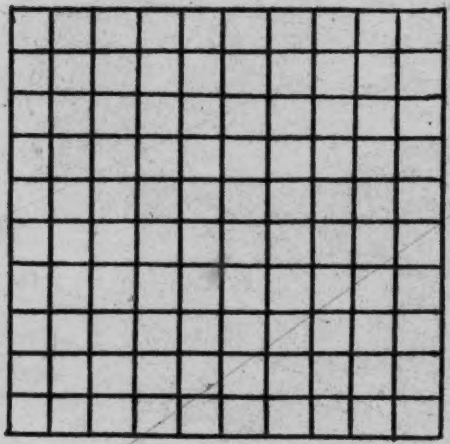
and the others as the importance of a redefinitive ability necessary to break from the traditional straight line cuts for one-half.

APPENDIX C

THE CONSTRUCTED PROBLEM-SITUATIONS

THE PROBLEM-SITUATIONS

i) Sensitivity I - The Square



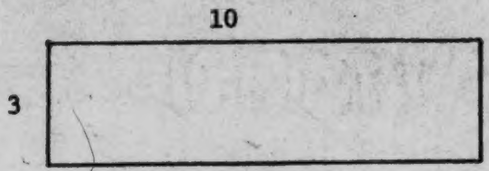
Cut the square in half. What are the shapes of the resulting halves? Draw diagrams to show your reasoning. Feel free to make comments about your thinking.

ii) Sensitivity II - Area-Increase

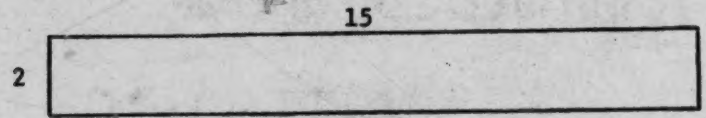
Two sides of a rectangle are increased by ten percent. How does the area of the new figure compare with that of the original figure? Draw a diagram. Indicate how you would go about solving this problem. Indicate procedures; it is not necessary to complete calculations.

iii) Redefinition I - The Board and the Hole

Cut the board into two equal pieces that will cover the hole completely. Show all your attempts including your incorrect ones.



BOARD

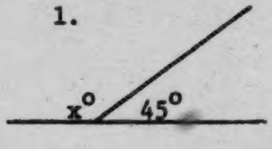


HOLE

iv) Redefinition IIA - Angles

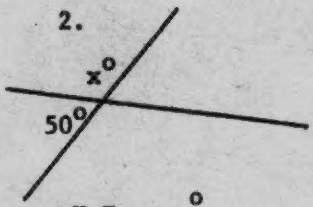
Find the measure of the indicated angles. Show your work. Drawings are not necessarily to scale.

1.



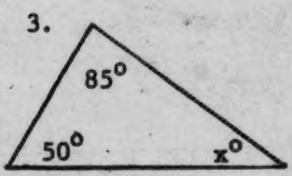
∠x = ____°

2.



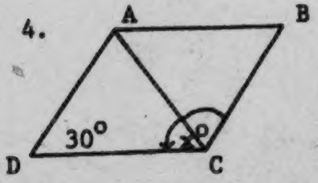
∠x = ____°

3.



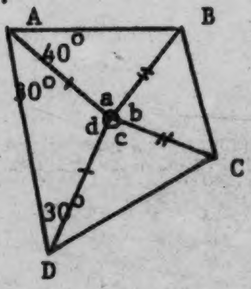
∠x = ____°

4.



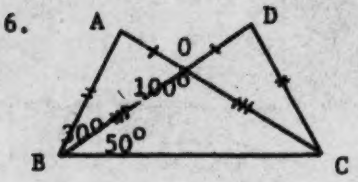
∠x = ____°

5.



AB ∥ BC
AO = OD
OB = OC
Find the measure of the angle formed by
∠(a+b+c+d) = ____°

6.

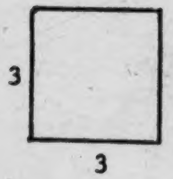


∠DCB = ____°

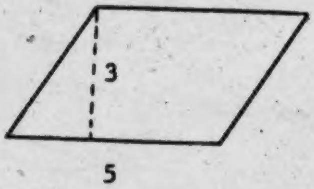
Redefinition IIB - Areas

Find the areas of the following figures. Show your work.

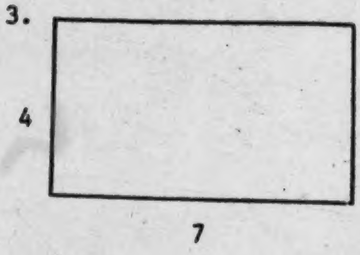
1.



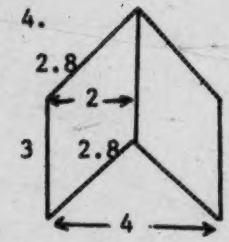
2.



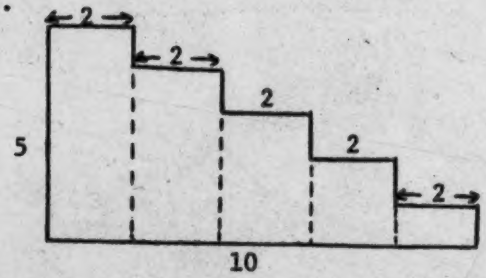
3.



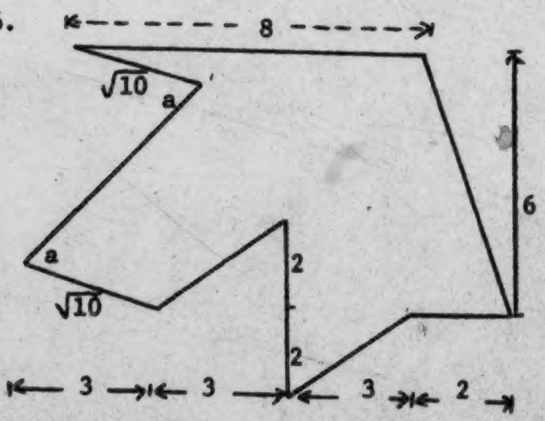
4.



5.



6.

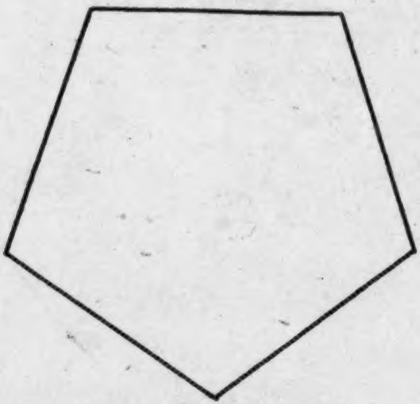


v) Conjecturing I - The Pentagon

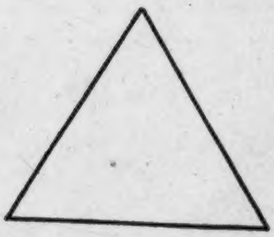
You are given the following shape. Make as many conjectures as you can about the given shape. One example is the following:

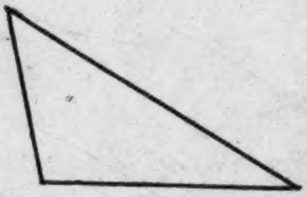
> A series of pentagons cannot cover a flat surface without leaving gaps unless the pentagons overlap.

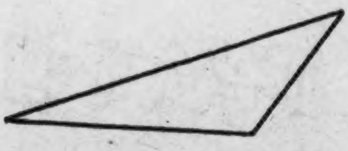
You can use this statement and vary it to make your own conjectures. Then make some of your own.

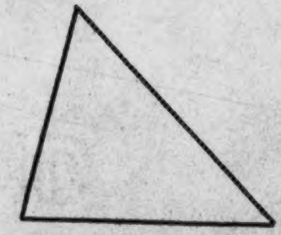


vi) Conjecturing II - The Triangles









The above is a sequence of triangles. Make some suggestions as to how these triangles are related. For example

The area of a triangle increases as the perimeter increases.

Use the conjecture to develop others if you wish. Then state some of your own.

vii) Verifying I - The Polygons

Consider the following statements. Prove each statement as best as you can. Show all your work.

(a) If the length of a side of a square is multiplied 5 times, the area is increased 25 times.

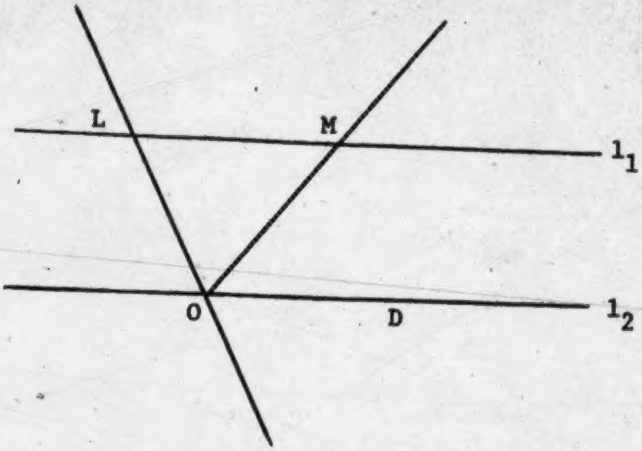
(b) If the length of a side of a triangle is tripled, the area is multiplied 9 times.

(c) If the length of a side of a pentagon is doubled, the area is multiplied 4 times.

(d) If the length of a side of a regular geometric figure is multiplied n times, the area is increased n^2 times.

viii) Verifying II - Parallel Lines

Prove each of the following statements. Show all your work. Diagrams are not necessarily drawn to scale.

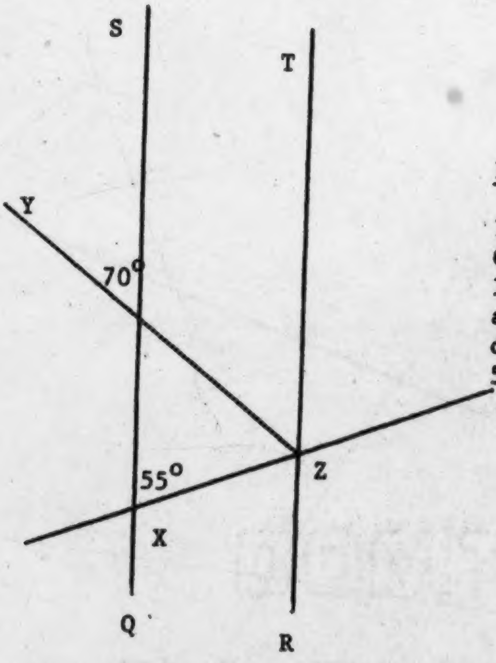


Part A - Abstract Argument

a. $l_1 \parallel l_2$

b. OM is a bisector of ∠LOD if and only if LM = LO.

Prove that this statement is true.



Part B - Specific Argument

If two lines are drawn parallel, QS, RT on the diagram, and the line YZ is drawn to meet QS at an angle of 70 degrees and XZ drawn so that it meets QS at 55 degrees, prove that XZ bisects ∠YZR.

APPENDIX D

THE RAW SCORES FOR FORTY-TWO INDIVIDUALS ON THE LORGE THORNDIKE, SCAT, MATHEMATICS ACHIEVEMENT, CREATIVITY, AND THE CONSTRUCTED TEST ITEMS

THE RAW SCORES FOR FORTY-TWO INDIVIDUALS ON THE LORGE THORNDIKE, SCAT, MATHEMATICS ACHIEVEMENT, AND CREATIVITY

	Age	Lorge Thorndike Verbal	Lorge Thorndike Non Verbal	SCAT Verbal	SCAT Quantitative	SCAT Percentile	Mathematics IX Department	Non Verbal Fluency	Non Verbal Flexibility	Non Verbal Originality	Verbal Fluency	Verbal Flexibility	Verbal Originality	Non Verbal Elab.
1	14-10	142	113	8	8	96	7	49	57	64	53	65	62	49
2	14-6	113	131	7	9	97	9	28	33	36	56	76	72	43
3	14-7	148	126	9	9	99	7	32	38	39	44	55	56	54
4	14-5	119	99	7	8	90	8	28	35	28	39	53	46	30
5	14-7	127	124	8	8	95	9	32	37	47	64	81	82	48
6	14-5	135	122	9	7	98	9	46	47	71	62	69	75	72
7	15-0	126	121	9	9	99	9	43	52	54	53	74	62	55
8	14-10	140	131	8	9	98	9	29	35	43	48	66	71	50
9	14-6	119	130	8	8	95	9	30	37	48	41	50	53	53
10	14-9	119	124	7	9	94	9	32	37	40	64	64	84	45
11	14-9	133	130	7	9	95	9	36	33	48	50	58	70	32
12	14-8	143	130	8	7	91	8	42	52	54	37	45	50	48
13		NA	NA	5	5	54	5	-	-	-	-	-	-	-
14	14-6	NA	NA	6	8	89	7	42	47	55	38	47	45	45
15	14-8	115	109	7	5	75	5	45	52	54	59	76	78	45
16	14-7	138	133	8	7	91	8	42	52	66	38	50	45	42
17	14-5	NA	NA	7	4	59	5	43	53	50	47	60	54	65
18	15-0	134	118	7	9	97	8	36	42	56	48	62	53	43
19	14-6	125	111	6	5	71	6	45	50	53	37	43	51	34
20	14-4	NA	NA	6	8	87	7	45	52	66	45	61	52	43
21	14-5	100	106	6	6	71	6	49	58	61	51	61	55	51
22	15-4	122	126	8	7	94	8	42	52	51	49	59	58	48
23	14-4	116	107	5	6	61	6	62	70	80	50	68	59	47
24	14-1	127	148	8	8	94	8	35	40	47	54	64	65	44
25	14-8	94	98	4	4	26	4	39	50	39	44	41	42	42
26	13-9	124	114	4	6	50	6	39	43	57	48	62	58	48
27	13-8	130	143	5	9	90	8	39	43	40	43	63	53	32
28	13-11	134	126	6	6	67	6	55	67	66	74	84	90	47
29	14-10	109	121	5	6	57	6	32	37	29	79	91	86	45
30	14-1	129	122	6	5	65	5	46	58	54	70	76	80	33
31	14-4	122	113	7	7	86	5	40	50	39	44	48	49	41
32	14-5	104	120	4	5	32	6	73	75	61	57	75	76	45
33	14-7	113	119	5	8	76	8	23	28	38	48	58	59	39
34	13-10	147	104	9	6	94	8	32	37	50	46	61	64	53
35	13-10	110	NA	8	8	96	9	42	45	60	83	90	100+	44
36	15-0	116	129	7	4	65	5	36	42	40	72	65	82	49
37	15-2	110	112	6	6	73	6	40	45	36	85	96	98	58

	Age	Lorge Thorndike Verbal	Lorge Thorndike Non Verbal	SCAT Verbal	SCAT Quantitative	SCAT Percentile	Mathematics IX Department	Non Verbal Fluency	Non Verbal Flexibility	Non Verbal Originality	Verbal Fluency	Verbal Flexibility	Verbal Originality	Non Verbal Elab.
38	13-9	124	113	7	5	75	6	39	43	43	41	55	48	48
39	14-5	112	110	5	5	50	3	39	43	51	53	61	71	47
40	14-11	120	130	6	7	83	7	42	47	38	55	65	60	41
41	14-3	111	106	6	6	67	5	48	45	64	67	74	78	59
42	14-9	133	116	9	6	91	6	52	55	75	64	75	87	51

THE RAW SCORES FOR FORTY-TWO INDIVIDUALS ON THE CONSTRUCTED TEST ITEMS

	Va	Vb	C1	C2	C3	Ca	Cb	Cc	S1	S2	S3	Sa	Sb	Sc	R1	RA	RB	V1	V2	V3	V4
1	0	5	4	4	4	5	5	4	8	8	4	2	1	0	3	5	3	5	2	0	4
2	4	5	3	2	4	6	5	4	8	6	4	2	1	0	3	3	3	6	6	6	6
3	2	0	3	2	1.5	5	3	1.5	4	2	1	2	1	0	3	4	6	3	3	0	0
4	0	1	3	2	0	4	1	2.5	4	2	0	2	2	3	0	2	2	6	1	4	4
5	0	5	2	2	0	3	1	0	6	4	2	3	2	4	2	4	1	6	5	5	5
6	2	5	3	3	2	3	2	2.5	5	3	1.5	2	2	4	0	0	2	2	2	0	1
7	0	5	9	4	3	7	3	3.5	8	5	3.5	2	1	0	1	4	2	2	2	2	0
8	2	2	3	2	1.5	3	2	0	5	3	1	4	3	4	3	3	2	3	3	3	4
9	0	3	2	2	1.5	4	1	2	7	6	4	1	1	0	0	2	2	1	1	1	0
10	4	5	3	3	.5	3	2	0	6	4	1	2	1	0	3	2	2	3	0	2	2
11	0	2	2	1	2	3	1	2	5	4	1	1	1	0	1	4	2	2	0	0	4
12	0	5	3	2	3	2	2	2	4	4	4	3	2	3	3	4	2	1	1	1	0
13	1	0	3	3	4	2	2	4	6	4	3	2	2	3	0	2	1	6	2	0	0
14	0	0	2	1	0	3	2	0	4	4	3.5	1	1	0	0	1	2	2	2	0	0
15	0	5	5	4	3	3	1	0	4	4	2.5	2	2	3	0	0	1	1	0	0	1
16	0	0	4	3	1.5	3	2	2	6	3	0	1	1	0	1	2	4	2	2	0	0
17	0	0	1	1	0	2	2	0	5	3	0	1	1	0	2	1	0	4	4	4	0
18	0	5	1	1	.5	4	2	2	5	4	1	2	1	0	0	4	2	6	2	4	4
19	1	0	4	3	1.5	4	2	0	1	1	0	0	0	0	0	3	2	0	4	0	0
20	0	5	0	0	0	2	1	1.5	6	5	2	4	3	4	0	2	6	4	2	0	0
21	0	0	5	3	4	5	2	0	6	4	3	0	0	0	0	1	2	0	0	0	0
22	0	5	3	2	1.5	4	1	2	8	4	2	2	2	4	0	2	2	3	0	0	0
23	1	0	1	1	1	2	1	0	7	5	2	0	0	0	0	2	2	3	0	0	0
24	3	5	2	2	1	2	1	0	6	4	1	2	1	0	3	2	2	3	3	3	0
25	0	0	4	2	.5	2	1	0	7	4	1	0	0	0	0	0	0	4	4	0	0
26	0	0	4	2	.5	0	0	0	8	3	1	2	1	0	0	1	2	1	0	0	0
27	0	5	2	1	1	4	1	0	5	5	2.5	2	1	0	2	3	2	2	2	2	0
28	0	4	5	2	.5	1	1	0	5	4	2.5	0	0	0	0	1	2	1	1	0	0
29	0	0	4	3	2	5	2	1.5	8	4	2.5	1	1	0	0	1	0	0	0	0	0
30	0	0	2	2	1.5	3	1	0	3	2	0	1	1	0	0	1	0	1	0	0	0
31	0	3	4	3	1.5	3	2	0	4	2	0	2	1	0	0	1	2	1	0	0	0
32	0	0	6	3	.5	4	2	0	5	3	0	0	0	0	0	0	1	1	0	0	0
33	0	0	4	2	1.5	4	2	2.5	3	3	3	2	2	3	0	0	0	2	1	0	0
34	1	1	4	3	3	6	3	4.0	7	5	3	2	2	3	0	2	1	3	2	0	2
35	3	3	3	2	1.5	4	4	2	6	5	2	2	2	3	2	4	2	5	5	5	5
36	0	0	3	2	3	2	2	2	3	2	0	1	1	0	0	0	2	1	1	0	0
37	0	0	4	2	.5	4	2	0	4	3	1	0	0	0	0	1	1	3	1	0	0
38	1	2	3	3	1.5	2	1	1	4	3	2	1	1	0	3	3	1	3	0	0	0
39	0	0	6	3	1.5	1	1	0	3	2	2.5	1	1	0	0	0	0	1	0	0	0
40	0	0	4	2	1	1	1	0	6	3	1	2	1	0	0	1	0	1	0	0	1
41	0	0	3	2	.5	1	1	2	10	4	3	1	1	0	0	1	3	2	2	0	1
42	0	5	4	2	1	2	2	2	5	5	4	1	1	0	0	4	3	2	2	0	1

LEGEND:

Va - Verifying IIA
Vb - Verifying IIB
C1 - Conjecturing I Fluency
C2 - Conjecturing I Variety
C3 - Conjecturing I Novelty
Ca - Conjecturing II Fluency
Cb - Conjecturing II Variety
Cc - Conjecturing II Novelty
S1 - Sensitivity I Fluency
S2 - Sensitivity I Variety
S3 - Sensitivity I Novelty
Sa - Sensitivity II Fluency
Sb - Sensitivity II Variety
Sc - Sensitivity II Novelty
R1 - Redefinition I
RA - Redefinition IIA
RB - Redefinition IIB
V1 - Verifying Ia
V2 - Verifying Ib
V3 - Verifying Ic
V4 - Verifying Id